Advanced Placement Calculus

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Contents

Chapter 1

Precalculus Review

1.1 Functions and their Graphs

1.1.1 Functions

All of the functions you will see in this course will be real-valued functions in a single variable. A function is real-valued if the input and output are real numbers as opposed to complex numbers. We will not work with any complex numbers in this course. Actually, the real title of this course is "The Calculus of a Single, Real Variable". In precalculus when you solved the equation $x^2 + 1 = 0$, you determined that the solutions were $x = i$ or $x = -i$. In this course, the equation $x^2 - 1 = 0$ has no solutions. Can calculus be done with complex numbers? Yes–but it is an entirely different course (one that you should definitely take in the future!) All of our functions will also be functions in a single variable. Nearly all the functions you have ever seen have been in a single variable. Algebraically, functions in a single variable look like $f(x) = x^3$ or $g(x) = \sin x$. Mathematically we can operate with functions in as many variables as we like. A function in two variables would look like $f(x, y) = x^2 + y^2$.

There are many ways to define a function. Pick up ten different mathematics texts and you will likely see ten different definitions–all saying the same thing but all phrased in different terms. In general, a function is a rule which pairs a member of an input set with a member of an output set.

Here are several definitions of a function:

- A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$ in a set b. (Stewart)
- A function from set D to a set R is a rule that assigns a unique element in R to each element in D. (Finney)
- A function f is a correspondence from a set A to a set B that associates with each element of a of A a unique element b of B. We denote this correspondence $\alpha(a) = b$ and we call b the image of a under α. (Larsen)

As you can see, functions are usually defined in terms of sets. Perhaps one of the clearest definitions is the following:

- A function is a set of ordered pairs in which no two distinct ordered pairs have the same first element. (Leithold)

Any set of ordered pairs is called a *relation*. Those sets of ordered pairs which meet any of the definitions listed above are functions. All functions are relations but not all relations are functions.

Throughout the course you will see functions presented in a variety of ways. Functions can be given algebraically, graphically, verbally, or in table form.

In terms of notation, there are several methods to express a function algebraically. Suppose we have a function that adds three to an input and then squares the sum. This function will most often be written $f(x) = (x+3)^2$. Another method you may have seen on occasion is $f : x \longrightarrow (x+3)^2$. The second technique makes a clearer distinction between the name of the function f, the input x, alnd the value of the function $(x+3)^2$. More formally, this relationship is written $\{(x, f(x))\,|\,f(x) = (x+3)^2\}$.

1.1.2 Domain

The domain of a relation or function is the set of all allowable inputs. Some functions have specific domains such as the natural logarithmic function. If $f(x) = \ln x$, the domain is $(0, \infty)$ because we cannot take the natural logarithm of zero or a negative number. For most other functions we encounter, we can find the domain by looking for variables in denominators and/or variables under even radicals. Finding the domain of a function is a straightforward procedure!

Consider the function $g(x) = \dfrac{1}{x-5}$. Since the denominator cannot be zero, $x \neq 5$, so the domain of g is $(-\infty, 5) \cup (5, \infty)$. The function $h(x) = \sqrt{x+8}$ has domain $[-8, \infty]$ because in order for h to exist, $x + 8 \geq 0$.

In the following example, we will demonstrate the proper way to show how the domain of a function is found. The days of just looking at a function a jotting down an answer are over. *All solutions require justification and nearly all justifications require words . . . your solutions should read like paragraphs.* This is something we will work on throughout the year.

Example 1

Find the domain of $f(x) = \sqrt{x^2 - x - 6}$.

$$f \exists \text{ when } x^2 - x - 6 \geq 0.$$

$$x^2 - x - 6 = 0 \longrightarrow (x-3)(x+2) = 0 \longrightarrow x = -2 \text{ or } x = 3.$$

$$\text{Now, } x^2 - x - 6 \geq 0 \text{ when } x \in (-\infty, -2] \cup [3, \infty)$$

$$\therefore \text{ the domain of } f \text{ is } (-\infty, -2] \cup [3, \infty).$$

Example 2

Find the domain of $p(x) = \dfrac{x+5}{x^2+5x+6}$.

$$p \nexists \text{ when } x^2+5x+6=0.$$

$$x^2+5x+6=0 \text{ when } (x+3)(x+2)=0 \longrightarrow x=-3 \text{ or } x=-2.$$

$$\therefore \text{ the domain of } p \text{ is } (-\infty,-3)\cup(-3,-2)\cup(-2,\infty).$$

Notice two things about these examples. First, we usually state the domain of a function as an interval and secondly, when we deal with functions with radicals, we normally phrase our process in terms of when the function will exist but when dealing with denominators, we phrase our process in terms of when the function does not exist. We find the values of x which we are going to eliminate.

In Example 1 we began our solution with "f exists when ..." but in Example 2 we began with "p does not exist when ...".

In Example 2, if we began with "p exists when ..." we would have had to say "... when $x^2+5x+6\neq 0$." The problem is that now many students will write

$$\begin{aligned} x^2+5x+6 &\neq 0 \\ (x+2)(x+3) &\neq 0 \\ x\neq -2 \text{ or } x &\neq -3 \end{aligned}$$

This may look fine but it is not proper mathematics. The steps used to solve a quadratic equation are valid for equations–statements of equality, not statements of "not equals". To avoid this predicament, we state solutions in terms which will allow us to work with "equals" instead of "not equals".

1.1.3 Range

The range of a function is the set of all outputs. Why not "allowable" outputs? Because there is not such thing! Outputs are not "allowable"–they are what they are. Range is generally more difficult to determine than domain–at least without using calculus. Once we learn about the derivative we will finally have a specific procedure to follow in order to determine range. For now, we are somewhat restricted. You can think of the range of a function as the "shadow" (the projection) of the function on the y-axis. Until we learn other techniques to determine range, we will primarily face two types of range problems. We will need to be able to find the range of functions such as $f(x) - \sqrt{x-4}$ and functions such as $g(x) = \dfrac{x^2-2x-15}{x-5}$. The first function involves an even radical. Because we use the $\sqrt{\ }$ symbol to denote the primary root, its value must always be greater than or equal to zero–so the range of f is $[0,\infty)$. g is an example of a rational, factorable function with a common term in the denominator and numerator. A rational function is a function of the form $R(x) = \dfrac{P(x)}{Q(x)}$ where both P and Q are polynomials. Note that g can be written $g(x) = \dfrac{(x-5)(x+3)}{x-5}$. Since we cannot divide by zero, $x \neq 5$. Now *imagine* the $x - 5$ term disappearing from the numerator and denominator. Now, if $x = 5$, the value of the "imagined" function would be 8. Therefore, since $x \neq 5$, $g(x) \neq 8$. The range of g is $(-\infty, 8) \cup (8, \infty)$.

Example 3

Find the range of $f(x) = \dfrac{x^2 - 9}{x - 3}$.

$$\text{Since } f(x) = \frac{(x-3)(x+3)}{x-3} \text{ and since } x \neq 3 \longrightarrow f(x) \neq 6.$$

$$\text{Therefore the range of } f \text{ is } (-\infty, 6) \cup (6, \infty).$$

1.1.4 Graphs of functions

We will spend a great deal of time in this course graphing functions. In fact, one of the most important and fascinating areas of calculus is its application to sketching curves. For now there are a few essential ideas we need to know and remember–and the first has to do with labeling. An arbitrary point on a curve can always be labeled in one of three ways. Consider the function $f(x) = x^3$. Any point on this curve can be labeled (x, y) or (x, x^3) or $\left(y^{1/3}, y\right)$. Knowing how to label arbitrary points in this manner allows us to work problems in terms of x and y, in terms of x only or in terms of y only.

You will also need to know how to quickly sketch the graphs of some basic functions without using a calculator. This list is not long but we need to be able to sketch: $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = x^3$, $f(x) = \ln x$, $f(x) = e^x$, $f(x) = \dfrac{1}{x}$, any linear function and any factorable quadratic.

In addition to the list above, we need to be able to handle vertical and horizontal translations. We will address translations in detail in the next section. The basic idea is that if we know what the graph of $f(x) = \dfrac{1}{x}$ looks like, we can easily know what the graph of $g(x) = \dfrac{1}{x-3} + 4$ looks like. The graph of g is the graph of f shifted three units to the right and four units up.

A few words about quadratics before we move on. Any quadratic equation can be manipulated to look like $y - k = a(x - h)^2$ or $a(y - k)^2 = x - h$ by completing the square. If we need to sketch the graph of a quadratic, we first rewrite it in one of these two forms. This process usually requires that we complete the square. Completing the square is one of our essential algebra skills that we will use over and over again all year long.

1.1.5 Piecewise functions

You are likely to see more piecewise functions this year than in any other previous mathematics course. For now, you need to be able to sketch simple piecewise functions and find function values. One of the most common piecewise functions involves absolute value. All functions that involve absolute value are piecewise functions and we need to be able to write them as such. Consider $f(x) = |x+5|$. To write this as a piecewise function, we first need to know the "breaking point"–the point where the graph of the function has a cusp. To determine this point, set the expression inside the absolute value bars equal to zero and solve. For $f(x) = |x+5|$, we set $x+5 = 0$ and obtain $x = -5$. We can now rewrite f as follows:

$$f(x) = \begin{cases} x+5 & \text{for} \quad \text{if } x \geq -5 \\ -x-5 & \text{for} \quad \text{if } x < -5 \end{cases}$$

Now that the absolute value bars have been removed, the function is in a more usable form. Throughout the course, whenever you are faced with a function involving absolute value, you will first rewrite it as a piecewise function.

1.1.6 Odd and even functions

A quick review of a couple of definitions should be enough to bring you up to speed.

If $f(x) = f(-x)\ \forall\ x,\ f$is even. Even functions are symmetric with respect to the x-axis.

If $-f(x) = f(-x)\ \forall\ x,\ f$is odd. Odd functions are symmetric with respect to the origin.

Typically, one of the first times students run into difficulty with "presentation" is with problems involving odd or even functions. Please take a close look at how these problems are presented.

Example 4

Determine if the following function is odd, even, or neither: $f(x) = x^5 + x$.

$$\text{Since } f(x) = x^5 + x \longrightarrow f(-x) = -x^5 - x \text{ and } -f(x) = -x^5 - x.$$
$$\text{Since } f(-x) = -f(x)\ \forall\ x, f \text{ is odd }.$$

Notice that we did not end the problem by simply stating, "f is odd". If we did, the reader of our problem is likely to ask, "Why?". You must state the definition as part of your answer!

Example 5

Determine if the following function is odd, even or neither: $g(x) = x^3 - 5x^2$.

$$\text{Since } g(x) = x^3 - 5x^2 \longrightarrow g(-x) = -x^3 - 5x^2 \text{ and } -g(x) = -x^3 + 5x^2.$$
$$\text{Since } g(x) \neq g(-x) \; \forall \; x, g \text{ is not even.}$$
$$\text{Since } -g(x) \neq g(-x) \; \forall \; x, g \text{ is not odd.}$$
$$\text{Therefore } g \text{ is neither odd nor even.}$$

Note that we did not simply find $g(-x)$ and $-g(x)$ and then write "neither" as an answer. We started by rewriting the function we are going to work with, we found $g(-x)$ and $-g(x)$ and then we clearly and completely stated our conclusion. It may take some time but you will all get very good at this with practice.

The value of knowing whether a function is odd or even is in the symmetry. If you are analyzing a function and can determine that it is even, you only need to analyze "half" of the function–the rest of the function behaves in a symmetric manner on the other side of the y-axis.

1.1.7 Operations on functions

Functions are mathematical objects that can be added, subtracted, multiplied, and divided. In addition functions can be *composed*. The four basic operations are simple enough but let's look at a few examples of composition and finding the domain of a composition.

Example 6

Given $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x-2}$, find $f \circ g$ and its domain.

<u>Finding $f \circ g$</u>

$$f(g(x)) = f\left(\frac{1}{x-2}\right) = x - 2.$$

<u>Finding the domain of $f \circ g$</u>

Although $x - 2 \; \exists \; \forall \; x$, the domain of g is $(-\infty, 2) \cup (2, \infty)$ and so the domain of $f \circ g$ is $(-\infty, 2) \cup (2, \infty)$

Important note: The domain of a composition must be a subset of the domain of the "inside" function. Suppose we composed two functions and the expression resulting from the composition exists on $(-\infty, 9) \cup (9, \infty)$ and the domain of the "inside" function is $(-\infty, 3) \cup (3, \infty)$. The domain of the composition will be $(-\infty, 3) \cup (3, 9) \cup (9, \infty)$.

You must be careful with your presentation here. Consider Example 6. A common student error is to start the domain discussion by saying that, "$f \circ g$ exists for all x...". This almost always leads to a contradiction in the conclusion. If a student says, "$f \circ g$ exists for all x..." and then concludes by saying that the domain of $f \circ g$ is $(-\infty, 2) \cup (2, \infty)$, they have contradicted themselves. You cannot say that a function exists for all x and then say that its domain is something different than all x. The way to avoid this error is to begin all discussions of the domain of a composition with the expression that is the result of the composition. Do not call it $f \circ g$. Instead, first state the interval or intervals where the expression exists, then state the domain of the "inside" function, intersect the two and end with "therefore the domain of $f \circ g$ is...".

Also note that this problem consists of two parts–finding the composition and then finding the domain–and each part of the solution is titled appropriately. When working on multi-part problems, you must tell the reader what you are about to do. A reader should never have to read your mind to determine what part of a problem is being done. Presentation counts.

Example 7

Given $f(x) = x + 2$ and $g(x) = \dfrac{1}{x}$, find $f \circ g$ and its domain.

Finding $f \circ g$

$$f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1+2x}{x}$$

Domain of $f \circ g$

$\dfrac{1-2x}{x}$ does not exist when $x = 0$ and the domain of g is $(-\infty, 0) \cup (0, \infty)$, thus the domain of $f \circ g$ is $(-\infty, 0) \cup (0, \infty)$.

1.2 Types of Functions, Translations and Scaling

1.2.1 Introduction

In this section we will first review a few families of functions. Throughout the course you will hear statements like "If f is a polynomial function...", or, "For rational functions...", and it will be important that you immediately know the form and basic behavior of the function that is being talked about. There are theorems which will only apply to specific types of functions so when you hear "rational function", you need to know exactly what a rational function is.

We will also address translations and scalings. The graph of a function is translated if it is moved horizontally or vertically. Scaling involves a function being compressed or stretched horizontally or vertically. Knowledge of translations and scalings allows us to quickly sketch graphs of a wide variety of functions. For example, if you know what $y = \sqrt{x}$ looks like, and you understand translations, you will immediately know what the graph of $y = \sqrt{x-7}$ or $y = 3 + \sqrt{x}$ looks like.

1.2.2 Types of functions

Constant Functions

- Constant functions are functions of the form $f(x) = c$ where c is a constant.
- The domain of any constant function is the set of all real numbers.
- The graph of any constant function is a horizontal line.
- Examples: $f(x) = 8,\ f(x) = e,\ g(x) = -1$

Power Functions

- Power functions are functions of the form $f(x) = x^a$ where a is a real number constant.
- If $a = -1$, the graph of the function will be a hyperbola.
- If $a = \dfrac{1}{n}$ where n in a positive integer, the function is a root function.

Polynomial Functions

- Polynomial functions are functions of the form $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$.
- The domain of a polynomial function is the set of all real numbers.
- For $n = 1$ f is a linear function.
- For $n = 2$ f is a quadratic function.
- For $n = 3$ f is a cubic function.
- For $n = 4$ f is a quartic function.
- For $n = 5$ f is a quintic function.

Rational Functions

- Rational functions are functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$ where both P and Q are polynomials. This is a very important definition! For some reason, many students think that any function which has variables in the numerator and denominator is a rational function. This is not true. The function $f(x) = \dfrac{x-4}{\sqrt{x+8}}$ is not a rational function because the denominator is not a polynomial.
- The domain of a rational function is the set of all reals such that $Q(x) \neq 0$.

Algebraic Functions

- Algebraic functions are functions that are constructed by performing algebraic operations (addition, subtraction, multiplication, division and taking roots) on polynomials. The function $f(x) = \dfrac{x-4}{\sqrt{x+8}}$ is an algebraic function. All rational functions are also algebraic functions…but not all algebraic functions are rational.
- We will spend a great deal of time in this course learning how to analyze algebraic functions.

Trigonometric Functions

- We'll do a complete review of these functions in a later section. Let it suffice to say that we *love* the trigonometric functions! Why? Because they are periodic. It's their periodicity that makes them predictable and easy to work with. For the purposes of this course, you do not actually need to know much trigonometry…but the part you need to know you need to know well.

Exponential Functions

- Exponential functions are functions of the form $f(x) = a^x$, where a is a positive constant. Do not confuse exponential functions with power functions! We will study exponential functions in great detail later in the course. Eventually exponential functions will become your absolute favorite family of functions.

Logarithmic Functions

- Logarithmic functions are functions of the form $f(x) = \log_a x$, where a is a positive constant. Logarithmic functions are the inverse of the the exponential functions and will be studied in detail when we address exponential functions.

1.2.3 Vertical and horizontal translations

You've spent time already in your precalculus courses studying vertical and horizontal translations. In the table below we have listed the various translations which can be performed on a function. Make sure you know and understand these translations–they will make your work easier later on. We will be sketching thousands of functions this year and the better you understand translations, the easier the course will be.

> Suppose $c > 0$
> The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted c units up.
> The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted c units down.
> The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted c units to the left.
> The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted c units to the right.

Here are some examples of translations:

- The graph of $f(x) = \dfrac{1}{x-6}$ will be the graph of $f(x) = \dfrac{1}{x}$ translated 6 units to the right.
- The graph of $f(x) = x^3 - 4$ will be the graph of $f(x) = x^3$ translated 4 units down.
- The graph of $h(x) = \ln(x + 2)$ will be the graph of $h(x) = \ln x$ translated 2 units to the left.

1.2.4 Scaling (stretching and compressing)

Scaling is sometimes more difficult to talk about than translations. This is due to the confusion caused by terms like "compressed vertically" vs. "stretched horizontally" and "stretched vertically" vs. "compressed horizontally". Visually these pairs of expressions seem to mean the same thing…but they are different. It is sometimes easiest to see the differences if we examine a function like $f(x) = \sin x$. If you look at the graph of $f(x) = 2\sin x$ you will notice that the amplitude is now 2 (the graph is "taller" but the zeros have not changed. This is a vertical stretch and is the result of replacing $y = f(x)$ with $y = cf(x)$. Now compare $f(x) = \sin x$ to $f(x) = \sin 2x$. The graph of $f(x) = \sin 2x$ is the same "height" as $f(x) = \sin x$ but the zeros have changed because the period changed. This is a horizontal compression and is the result of replacing $y = f(x)$ with $y = f(cx)$. The details are shown in the table below.

> Suppose $c > 0$
> The graph of $y = c\,f(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of c.
> The graph of $y = \dfrac{1}{c} f(x)$ is the graph of $y = f(x)$ compressed vertically by a factor of c.
> The graph of $y = f(c\,x)$ is the graph of $y = f(x)$ compressed horizontally by a factor of c.
> The graph of $y = f\left(\dfrac{1}{c}\,x\right)$ is the graph of $y = f(x)$ stretched horizontally by a factor of c.

1.2.5 Reflections

In addition to the translations and scalings discussed above, graphs of functions can be reflected about axes.

> The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis.
> The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y-axis.
> The graph of $y = |f(x)|$ is the graph of $y = f(x)$ in which all the portions of the curve below the x-axis are reflected across the x-axis.

1.3 Trigonometry Review

1.3.1 Introduction

The six trigonometric functions are examples of transcendental functions. If you are like most students, as soon as you hear "trig" your mouth dries up, your brain shuts down and you feel like you're about to be sent back to topic you never really understood in the first place. Not to worry. Relax. There is no panic in AP Calculus! The trigonometric functions are actually very well-behaved functions. The phrase "well-behaved" is an actual phrase used by mathematicians to describe functions that are predictable, smooth functions. The trigonometric are among the most well-behaved functions. This is because they are all periodic. If you know how they behave over one period, you know how they behave everywhere! The amount of trigonometry that you need to know to be successful in this course is minimal ... but you need to know it! You need to be able to recall it instantly–otherwise, problems that are actually trivial will become "undoable". Basically, you need to be able to use reference angles (or the unit circle), solve simple trigonometric equations and you will need to memorize a dozen or so identities.

1.3.2 Radians, radians, radians

Every problem that involves trigonometry that you encounter this year must be done in radians. In mathematics the trigonometric functions are defined in terms of radians. Let's get started by finally understanding what a radian is and why we use them.

First of all, radians are real numbers, degrees are not. Degrees are a relic from the Babylonians who had a base 60 number system and decided that it would be nice to divide a circle into 360 equal parts–mostly because 360 is divisible by 60 and a host of other integers. A degree is an arbitrary unit and is not based on any measurement. That being said, we will at time make use of degrees in the process of solving a problem but we will give our answers in radians. A radian however, is a real number. A radian is the measure of an arc of a circle. Rather than measuring an angle by how spread apart the sides are, we will measure angles by the length of the arc the angle subtends on a circle. This is why so many familiar angles involve π. The circumference of a circle is given by $C = 2\pi r$. We normally use a unit circle ... a circle of radius 1. (The radius could be one foot, one inch or one mile.) This means that the circumference of the unit circle is 2π units. If the radius was one foot, the circumference would be 2π feet (about 6.283 feet). A right angle would take us 1/4 of the way around the circle so a right angle becomes $\pi/2$ radians. Similarly,going halfway around the circle means we would travel π units.

Identities you need to know (memorized) are listed below:

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\csc x = \frac{1}{\sin x}$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\sin^2 x + \cos^2 x = 1$$
$$1 + \cot^2 x = \csc^2 x$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$
$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 1 - \sin^2 x$$

1.3.3 Conversion between radians and degrees

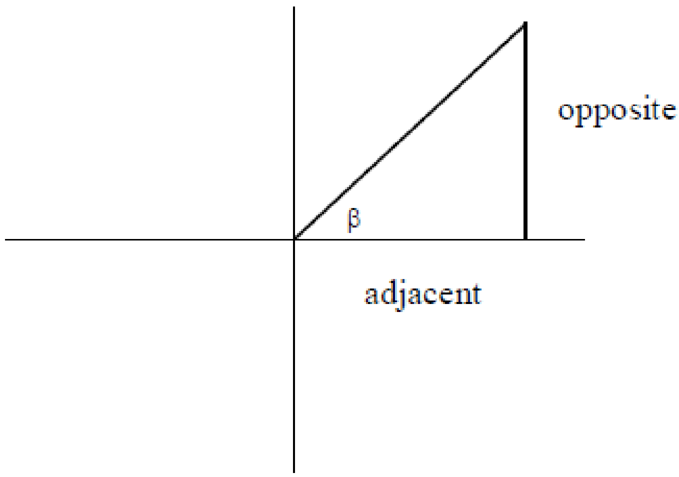
It is useful, especially when using reference angles, to do a problem in degrees instead of radians (even though we always put our final answers in radians–and always use radians when using the calculator.) To covert from degrees to radians, multiply the degrees by $\pi/180$. To convert from radians to degrees, multiply the radians by $180/\pi$.

1.3.4 Standard values

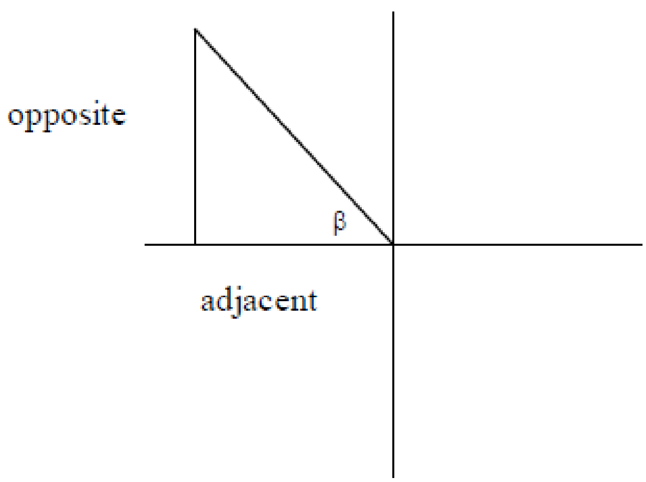
You will need to have memorized (or be able to quickly calculate) the value of the six trigonometric functions for any angle that is a multiple of 30 or 45 degrees. If you have trouble memorizing the values, simply use reference angles for those divisible by 30 and 45 and use the unit circle for angles that are multiples of 90. All you need to know to use reference angles is the ratio of the sides of a 30-60-90 and a 45-45-90 triangle.

1.3.5 Using reference angles

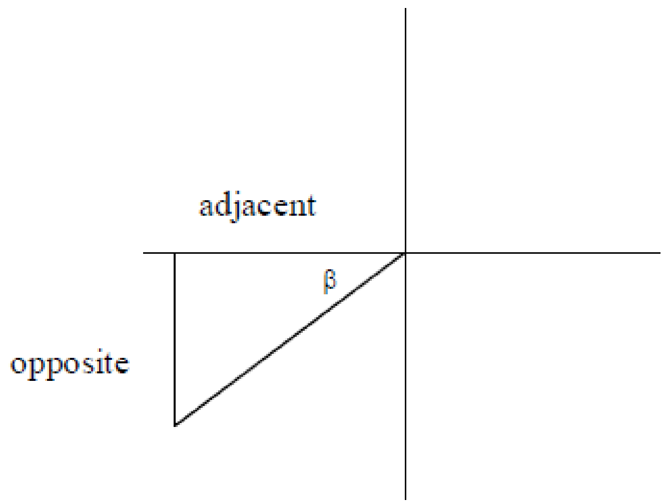
To use reference angles, first convert your angle measurement into degrees. If the angle is in the first quadrant, the reference angle and triangle will look like this:



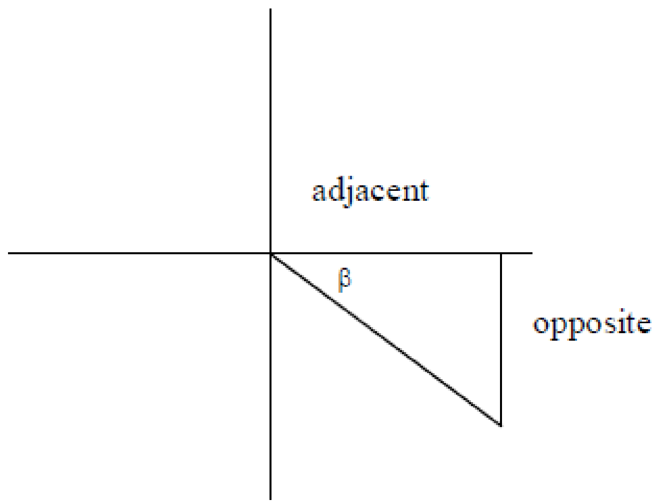
If the reference angle is in the second quadrant, the angle and triangle will look like this:



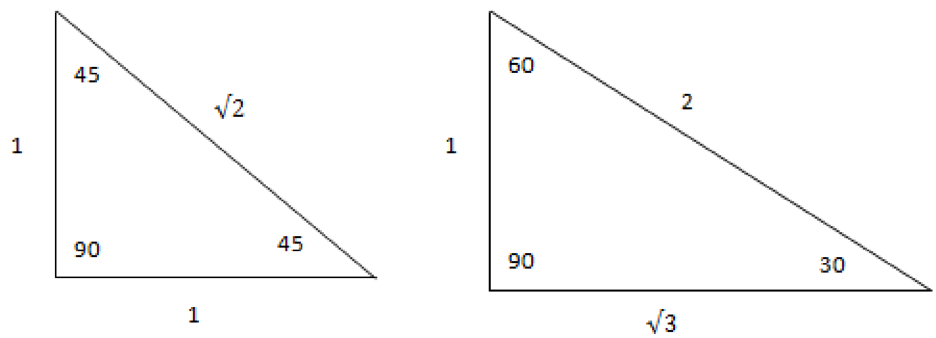
If the reference angle is in the third quadrant, the angle and triangle will look like this:



If the reference angle is in the fourth quadrant, the angle and triangle will look like this:



After drawing our reference angle and triangle, we label the sides of the triangle using one of the two special triangles illustrated below. Depending on which quadrant the triangle is in, one or both of the sides will be "negative". A distance cannot actually be negative, but when the triangle is drawn on the coordinate system, we denote sides which are drawn in a negative direction as negative.



Now that we have our angle converted and our triangle drawn and labeled, we make use of the following relationships:

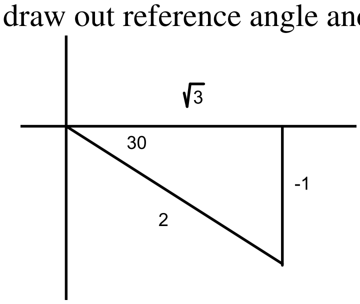
$$\sin x = \frac{opposite}{hypotenuse}$$
$$\cos x = \frac{adjacent}{hypotenuse}$$
$$\tan x = \frac{opposite}{adjacent}$$
$$\cot x = \frac{adjacent}{opposite}$$
$$\sec x = \frac{hypotenuse}{adjacent}$$
$$\csc x = \frac{hypotenuse}{opposite}$$

Example 8

Find $\sin\left(\frac{11\pi}{6}\right)$.

We begin by converting our angle to degrees: $\frac{11\pi}{6} = 330^\circ$

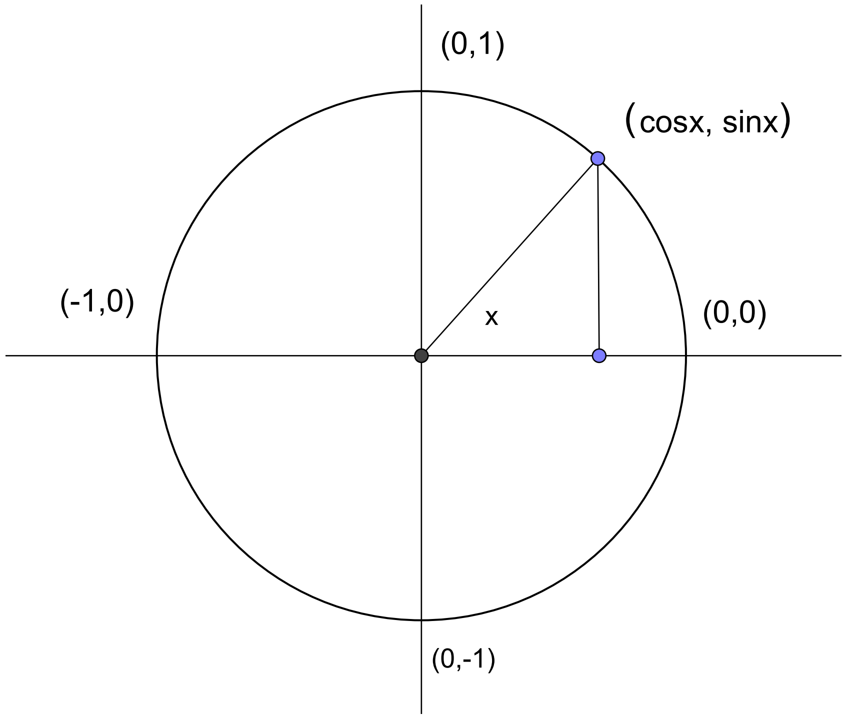
Next we draw out reference angle and triangle:



Since sine is the opposite side divided by the hypotenuse, $\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$

1.3.6 Using the unit circle

If we need to find the value of a trigonometric function that is a multiple of 90 we will use the unit circle. Any point on the unit circle can be labeled $(\cos x,\ \sin x)$ as seen in the diagram below.



Examle 9

Find $\sin \pi$.

π radians is 180 degrees.
Sine is given by the second number in the coordinate pair, so $\sin \pi = 0$.

More often than not you will be asked to find an angle whose trigonometric value is given. This happens all the time when we solve equations involving trigonometric functions. Instead of being given an angle and finding out what the sine of the angle is, for example, you will be given the sine of the angle and then be asked to find the angle or angles.

Example 10

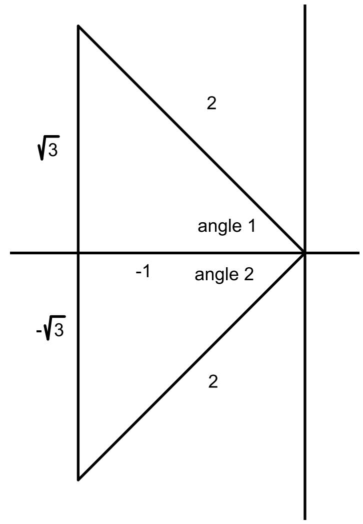
Find the angle(s) in the interval $[0, 2\pi]$ whose cosine is $-\frac{1}{2}$ Note: This is the same as solving the equation $\cos x = -\frac{1}{2}$ on $[0, 2\pi]$.

We need a reference angle and triangle the that the cosine of the reference angle is $-\frac{1}{2}$.
We know that the cosine of an angle is the adjacent side divided by the hypotenuse.

We leave the negative sign with the numerator (the adjacent side)since, by convention, the hypotenuse is always positive.

Adjacent sides are always drawn along the x-axis, opposite sides are drawn perpendicular to the x-axis.

Below is the diagram. Notice that there are two reference angles that have a cosine equal to $-\frac{1}{2}$



The third side of these two triangles is $\sqrt{3}$ and the triangle is a 30-60-90 with angle 1 and angle 2 both equal to 60 degrees.

Now, we can say that the two angles in $[0, 2\pi]$ whose cosine is $-\frac{1}{2}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Note that if the original interval was $[-2\pi, 0]$, the answers would be $-\frac{4\pi}{3}$ and $-\frac{2\pi}{3}$. We get these angles by moving in a negative direction (clockwise).

1.3.7 Solving Trigonometric Equations

We will rarely deal with any but the most simple trigonometric equations.

Example 11

Solve $\sin 2x = 0$ on $[0, 2\pi]$.

First we ask ourselves, "When is the sine of *any* angle equal to zero?"

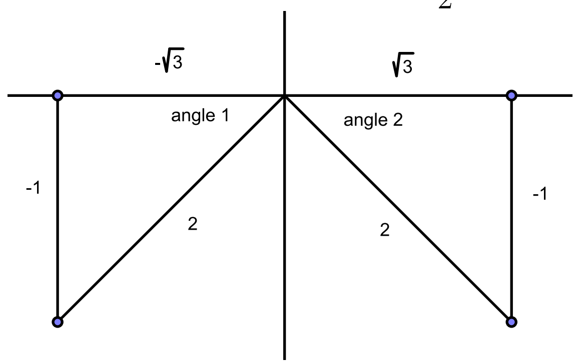
Using the unit circle we see that the sine will be zero when the angle is 0 or π so $\sin 2x = 0$ when $2x = 0$ or $2x = \pi$ or $2x = 2\pi$ or $2x = 3\pi$ or $2x = 4\pi$

$$\therefore x = 0 \text{ or } x = \frac{\pi}{2} \text{ or } x = \pi \text{ or } x = \frac{3\pi}{2} \text{ or } x = 2\pi.$$

Example 12

Solve $\sin 2x = -\dfrac{1}{2}$ on $[0, 2\pi]$.

First we determine when the sine of any angle is equal to $-\dfrac{1}{2}$. We do this by drawing a diagram.



The third side of these two triangles is $\sqrt{3}$ and the triangle is a 30-60-90 with angle 1 and angle 2 both equal to 30 degrees.

These two reference angles correspond to $\dfrac{7\pi}{6}$ and $\dfrac{11\pi}{6}$.

We can now say that $\sin 2x = -\dfrac{1}{2}$ when

$$2x = \frac{7\pi}{6} \text{ or } 2x = \frac{11\pi}{6} \text{ or } 2x = \frac{19\pi}{6} \text{ or } 2x = \frac{23\pi}{6}$$

$$\therefore x = \frac{7\pi}{12} \text{ or } x = \frac{11\pi}{12} \text{ or } x = \frac{19\pi}{12} \text{ or } x = \frac{23\pi}{12}$$

1.4 Inequalities and Absolute Value

1.4.1 Introduction

In this section we will discuss techniques used to solve a wide variety of problems involving inequalities, absolute value or both. we will encounter problems of this nature throughout the course so it is essential that we can handle them with ease.

Whenever the answer to a problem is an interval, we will use interval notation. The table below should refresh your memory.

Inequality Notation	Interval Notation
$a < x < b$	(a, b)
$a \leq x < b$	$[a, b)$
$a < x \leq b$	$(a, b]$
$a \leq x \leq b$	$[a, b]$
$x > a$	(a, ∞)
$x \geq a$	$[a, \infty)$
$x < a$ or $x > b$	$(-\infty, a) \cup (b, \infty)$
$x < a$ or $x \geq b$	$(-\infty, a) \cup [b, \infty)$
$x \leq a$ or $x > b$	$(-\infty, a] \cup (b, \infty)$
$x \leq a$ or $x \geq b$	$(-\infty, a] \cup [b, \infty)$
All the reals	$(-\infty, \infty)$

The table does not include all the possible variations, but you get the idea. Note that an open parentheses is always used with the infinity symbol.

1.4.2 A special note about solving equations involving rational expressions

You've solved equations involving rational expressions since your first algebra course. You may have been taught to multiply both sides of the equation or inequality by a common denominator–thus "eliminating" the denominator. Consider the following equation which we will solve using this technique.

$$\frac{5x}{x+6} = 2$$

$$(x+6)\left[\frac{5x}{x+6}\right] = 2(x+6)$$

$$5x = 2x + 6$$

$$3x = 12$$

$$x = 4$$

This is, in fact, the correct solution so what's the issue you ask? Watch what happens when we try the same technique with another equation.

$$x + 3 = \frac{-2x^2 + 7x - 3}{x - 3}$$

$$(x-3)\,[x+3] = \left[\frac{-2x^2 + 7x - 3}{x - 3}\right](x-3)$$

$$x^2 - 9 = -2x^2 + 7x - 3$$

$$3x^2 - 7x - 6 = 0$$

$$(3x+2)(x-3) = 0 \;\longrightarrow\; x = -\frac{2}{3} \text{ or } x = 3$$

Notice that $x = 3$ cannot be a solution because it would make the denominator on the left side equal to zero. Multiplying by a variable expression has induced an extraneous solution!

In this course, many of our problems will be loaded with subtleties and nuances. We will have enough to think about without having to worry about whether we've introduced extraneous solutions as part of solving the larger problem.

To avoid the possible introduction of extraneous solutions, we will use a different technique to solve *and and all* non-linear equations and inequalities.

In order to solve a non-linear equation or inequality we will

- Bring all terms to one side
- Find a common denominator to get a single fraction
- Set the numerator equal to zero to find solutions.

This technique eliminates any possibility of introducing extraneous solutions. It also allows us to easily determine where a particular expression fails to exist. Let's take a look at how it works. Consider the following:

$$\frac{x-3}{x+4} = 5$$

$$\frac{x-3}{x+4} - 5 = 0$$

$$\frac{(x-3)(1) - 5(x+4)}{x+4} = 0$$

$$\frac{-4x - 23}{x+4} = 0$$

$$-4x - 23 = 0 \;\longrightarrow\; x = -\frac{23}{4}$$

This may seem like a longer procedure for some problems, and it may very well be longer, but it eliminates any need to check for extraneous solutions.

1.4.3 Inequalities without absolute value

You've been solving inequalities for several years now and hopefully you're comfortable solving simple linear inequalities like $2 + 3x < 5x + 8$ so we'll move on to slightly more interesting examples.

Example 13

Solve: $\frac{7}{x} > 2$

$$\frac{7}{x} > 2$$

$$\frac{7}{x} - 2 > 0$$

$$\frac{7 - 2x}{x} > 0$$

$$\therefore x \in \left(0, \frac{7}{2}\right)$$

The final answer was obtained by using a sign chart. The expression is equal to zero when $x = \frac{7}{2}$ and fails to exist at $x = 0$. These are the number we use to set up our chart.

Note on sign charts: As you remember, when making a sign chart you must consider both where the expression is equal to zero *and* where it fails to exist. We won't make as many sign charts as you did last year. We will only use them when we encounter more complicated expressions. For quadratics and factorable cubics we will simply find the zeros and draw a quick sketch. *A sign chart will never be a graded part of a solution!* You can even do them on scratch paper... I don't look at sign charts when I read your problems.

By the way... look at what happens if we don't use the correct technique and try multiplying both sides by x.

$$\frac{7}{x} > 2$$
$$7 > 2x$$
$$x < \frac{7}{2}$$

Note that this solution includes zero and all the negative numbers... but the original inequality does not exist at $x = 0$ and is not satisfied by any negative number.

Example 14

Solve: $\dfrac{x}{x-3} < 4$

$$\frac{x}{x-3} < 4$$

$$\frac{x}{x-3} - 4 < 0$$

$$\frac{x - 4(x-3)}{x-3} < 0$$

$$\frac{-3x + 12}{x-3} < 0$$

$$\therefore \ x \in (-\infty, 3) \cup (4, \infty)$$

Example 15

Solve: $2x - 1 < 5x + 5 < 15$

You've probably been taught two different techniques for solving combined inequalities. One technique is to operate on all three parts at the same time. The other technique is to separate the statement into two parts using the property of inequalities which states that if $A < B < C$ then $A < B$ and $B < C$. The problem is that some combined inequalities cannot be solved using the first technique. If you have variables in two or more parts of a combined inequality, chances are that when you try to operate on all the parts at the same time, you end up "chasing" the variable around. So, let's apply the property and solve.

$$2x - 1 < 5x + 5 < 15$$
$$2x - 1 < 5x + 5 \text{ and } 5x + 5 < 15$$
$$-3x < 6 \text{ and } 5x < 10$$
$$x > -2 \text{ and } x < 2$$
$$\therefore x \in (-2, 2)$$

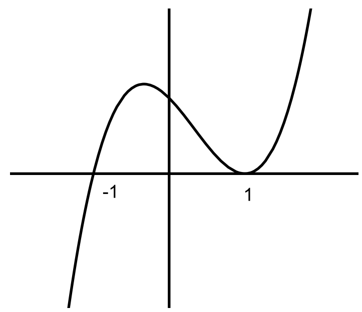
Example 16

Solve: $x^3 + 1 > x^2 + x$

Like all non-linear inequalities, our first step is to bring everything to one side, find out where it is equal to zero and where it fails to exist and then use a "1–2" chart or a quick sketch.

$$x^3 + 1 > x^2 + x$$
$$x^3 - x^2 - x + 1 > 0$$
$$(x + 1)(x - 1)(x - 1) > 0$$
$$\therefore x \in (-1, 1) \cup (1, \infty)$$

This answer can be obtained without a sign chart. Note that we have a positive cubic with zeros at $x = -1$ and $x = 1$. A quick sketch looks like this:



The curve is clearly above the x-axis on $(-1, 1) \cup (1, \infty)$. We had to throw out the $x = 1$ because the original problem is "less than" and not "less than or equal to".

1.4.4 Absolute value

Before we address inequalities involving absolute value, we should quickly review absolute value itself. There are several ways to define absolute value and which one we use depends on the situation.

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

$|x|$ is the distance from x to zero on the number line.

$$|x| = \text{max} x, -x$$

In Calculus we cannot perform any operations on expressions that are inside absolute value bars. That's why we have all the cases we use. . . to get rid of the absolute value. One critical skill you will need is the ability to rewrite any linear expression inside of absolute value bars without the absolute value. To do this, determine where the linear expression is equal to zero. The number tell us where the graph "breaks".

Example 17

Write $|x-5|$ without absolute value.

$$x-5=0 \longrightarrow x=5 \text{ so}$$
$$|x-5| = \begin{cases} x-5 & \text{for } x \geq 5 \\ 5-x & \text{for } x < 5 \end{cases}$$

Notice that $5-x$ is just a clean way to write $-(x-5)$

Example 18

Write $|x+3|$ without absolute value.

$$x+3=0 \longrightarrow x=-3 \text{ so}$$
$$|x+3| = \begin{cases} x+3 & \text{for } x \geq -3 \\ -x-3 & \text{for } x < -3 \end{cases}$$

Notice that $-x-3$ is just $-(x+3)$

Example 19

Write $|2x-9|$ without absolute value.

$$2x-9=0 \longrightarrow x=\frac{9}{2} \text{ so}$$
$$|2x-9| = \begin{cases} 2x-9 & \text{for } x \geq 9/2 \\ 9-2x & \text{for } x < 9/2 \end{cases}$$

Here are several important theorems dealing with absolute value which we use to solve inequalities involving absolute value.

$$|x| < a \iff x < a \text{ and } x > -a$$
$$|x| \leq a \iff x \leq a \text{ and } x \geq -a$$
$$|x| > a \iff x > a \text{ or } x < -a$$
$$|x| \geq a \iff x \geq a \text{ or } x \leq -a$$

As mentioned above, we often think of absolute value in terms of distance. Using this idea and the theorems above, let's consider the inequality $|x-3|<5$.

$$\begin{gathered}|x-3|<5\\ x-3<5 \text{ and } x-3>-5\\ x<8 \text{ and } x>-2\end{gathered}$$

Notice that the solution includes all the numbers between -2 and 8. The number halfway between these two numbers is 3. The distance from -2 to 3 is 5 and the distance from 3 to 8 is also 5 so the inequality $|x-3|<5$ describes all the x's that are within 5 units of 3.

Similarly, the inequality $|x-7|<9$ describes all the x's that are within 9 units of 7… or (-2,16).

We bring up this interesting way to read or translate these simple inequalities into English because very soon we will be discussing the definition of "limit" and if we can *read* the inequalities in the definition, the definition will make much more sense.

1.4.5 Solving inequalities involving absolute value

Again, we will consider several different types of inequalities involving absolute value–starting with the simpler and moving to the more complex.
Our first example involves absolute value on one side of an inequality.

Example 20

Solve: $|3x+2|>5$

$$\begin{gathered}|3x+2|>5\\ 3x+2>5 \text{ or } 3x+2<-5\\ 3x>3 \text{ or } 3x<-7\\ x>1 \text{ or } x<-\frac{7}{3}\\ \therefore x\in\left(-\infty,-\frac{7}{3}\right)\cup(1,\infty)\end{gathered}$$

Our next example involves absolute value on one side of an equation.

Example 21

Solve: $|3x+2|=5$

$$|3x+2|=5$$
$$3x+2=5 \text{ or } 3x+2=-5$$
$$3x=3 \text{ or } 3x=-7$$
$$x=1 \text{ or } x=-\frac{7}{3}$$
$$\therefore x \in \left\{1,-\frac{7}{3}\right\}$$

Note that this is not an interval. It is a solution set...a list of solutions.

Now, if the absolute value quantity is equal to a variable expression and not a number, we have to be more careful. The theorem below will help us.

$$|A|=B \longrightarrow A=B \text{ or } A=-B \text{ only if } B\geq 0$$

Example 22

Solve: $|5x-8|=3x+2$

First we determine the restriction on our answers.

$$3x+2\geq 0 \longrightarrow x\geq -\frac{2}{3}$$

Now,

$$5x-8=3x+2 \text{ or } 5x-8=-(3x+2)$$
$$x=5 \text{ or } x=\frac{3}{4}$$

Since both of these solution meet our restriction, they are both valid solutions.

Example 23

Solve: $|3x+4| = x+1$

First we determine the restriction on our answers.

$$x+1 \geq 0 \longrightarrow x \geq -1$$

Now,

$$3x+4 = x+1 \text{ or } 3x+4 = -(x+1)$$

$$x = -\frac{3}{2} \text{ or } x = -\frac{5}{4}$$

Since neither of these solution meet our restriction, there are no solutions.

Example 24

Solve: $|3x+4| = -4x-2$

First we determine the restriction on our answers.

$$-4x-2 \geq 0 \longrightarrow x \leq -\frac{1}{2}$$

Now,

$$3x+4 = -4x-2 \text{ or } 3x+4 = -(-4x-2)$$

$$x = -\frac{6}{7} \text{ or } x = 2$$

Since $x = 2$ does not meet our restriction, the only solution is $x = -\frac{6}{7}$.

Finally we run into the case where we have absolute value on both sides of an equation. We will use the following theorem to unlock this one.

$$|A| = |B| \longrightarrow |A| = B$$

Notice that since our original problem had the right side inside absolute value bars, there is no restriction on our answers!

Example 24

Solve: $|x+4| = |2x-6|$

First we rewrite the problem,

$$|x+4| = 2x-6$$

Now we solve using the usual absolute value definition.

$$x+4 = 2x-6 \text{ or } x+4 = -(2x-6)$$

$$x = 10 \text{ or } x = \frac{2}{3}$$

Since there are no restrictions on our answers, both of these solutions are valid.

Our final example will involve an inequality with absolute value and a non-linear expression.

Example 25

Solve $\left|\frac{6-5x}{x+3}\right| \leq \frac{1}{2}$.

$$\left|\frac{6-5x}{x+3}\right| \leq \frac{1}{2}$$

$$\frac{6-5x}{x+3} \leq \frac{1}{2} \text{ and } \frac{6-5x}{x+3} \geq -\frac{1}{2}$$

$$\frac{6-5x}{x+3} - \frac{1}{2} \leq 0 \text{ and } \frac{6-5x}{x+3} - \frac{1}{2} \geq 0$$

$$\frac{9-11x}{2x+3} \leq 0 \text{ and } \frac{15-9x}{2x+3} - \frac{1}{2} \geq 0$$

Both of these inequalities are solved using "1-2" charts. The two solutions are then intersected.

$$\therefore x \in \left[\frac{9}{11}, \frac{5}{3}\right]$$

1.4.6 Summary

When solving inequalities:

- If the problem involves only linear expressions, we solve it like we solve linear equations
  - variables to one side, constants to the other
  - if you multiply or divide by a negative, change the direction of the inequality
- If the problem involves a non-linear expression:
  - get everything on one side so we are comparing the expressions to zero
  - get a common denominator if you have variables in denominators

  - determine where your expression is equal to zero and and where it fails to exist and use a “1-2” chart if necessary to

- If the problem involves absolute value
  - set up the appropriate cases using definitions
  - remember that the case where $|A| < B$ and B is a variable expression requires that you find any restriction on your solution

Chapter 2

Limits

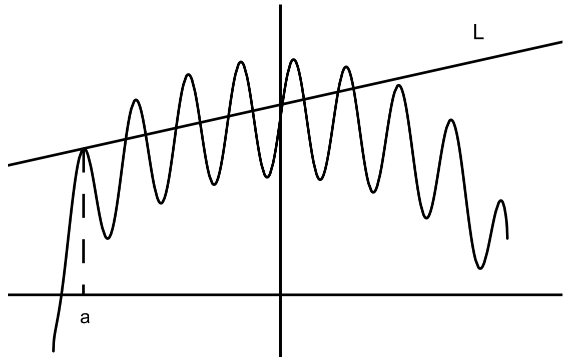
2.1 Tangents and the Velocity Problem

2.1.1 Introduction

This section is actually just a preview of what's to come... and a history lesson. The final development of Calculus in the 17th century was centered around several problems from both the applied and pure branches of mathematics. One problem involved finding a way to determine the instantaneous velocity of an object. Mathematicians realized that this was closely related to the problem of finding the slope of a tangent to a curve. The other problem was actually an ancient one... finding the exact volume of an arbitrary solid. Two thousand years ago, Archimedes worked on, and made significant progress in determining volumes of solids. We will actually use his method later on in the course. In this section we will discuss the velocity and tangent problem.

The techniques we will use in this section will not give us exact answers. We will be using fairly "ugly" methods to *estimate* the slope of an tangent to a curve. Later on, after we know how to work with "limits" we will be able to find the exact slope of a tangent to a curve at a given point.

We've used the word "tangent" several times now. In geometry, you studied tangents–primarily tangents to circles. In this course, when we speak of a line being tangent to a curve we mean tangent *at a point*. What the tangent line does once it leaves the neighborhood of the point does not matter to us. In geometry you learned that a tangent touches a circle in exactly one point. In calculus you will see many tangents that intersect curves... sometimes many times. Consider the graph on the next page.



The line L is tangent to the curve at point a. Notice that L intersects the curve at several other points.

2.1.2 Approximating the slope of a tangent–function in algebraic form

Let's try to write an equation for a line tangent to the graph of $f(x) = x^2$ at $x = 1$. A tangent line is, of course, just a line. All we ever need to write the equation of a line is a point and the slope. The point is easy. In this case since $f(1) = 1$, the point is $(1, 1)$.

We will not be able to find the exact slope of this tangent. Not yet. Not until we know about limits and derivatives. What we will do for now is find ways to approximate the slope of a tangent.

Before we work on finding the slope of a tangent line, we need to know what a secant line is. A secant line is a line that intersects, but is not tangent to, a curve. It is essentially a line that intersects a curve in two or more places. To begin estimating the slope of the tangent we will find the slope of a secant that passes through $(1, 1)$ and another point Q on the curve close to $(1, 1)$. We will continue to find the slope of this secant line as Q gets closer and closer to $(1, 1)$. Another way to say this is we will let Q approach$(1, 1)$. As you can likely see, there are two ways to get closer to $(1, 1)$... from the right and from the left–we'll do both.

We'll start by approaching $(1, 1)$ from the right. This means that we will use values of x like 1.5, then 1.4, then 1.3 and so on. Now we could find the y-values for each of these x's and then calculate the slope of the secant through $(1, 1)$ and each of these points. This will be time consuming and tedious. Instead, let's put an arbitrary point Q on the curve. It's coordinates will be (x, x^2). Now the slope of the secant through $(1, 1)$ and (x, x^2) will always be $\dfrac{x^2 - 1}{x - 1}$ The table below shows the resulting secant line slope for several values of x.

x	$\dfrac{x^2-1}{x-1}$
1.500	2.500
1.100	2.100
1.010	2.010
1.001	2.001

Notice that as Q gets closer and closer to $(1,1)$ from the right, the slope of the secant line gets closer and closer to 2. We will soon learn that this is written $\lim_{x \to 1^+} \frac{x^2-1}{x-1} = 2$.

We will now approach $(1,1)$ from the left.

x	$\frac{x^2-1}{x-1}$
.500	1.500
.900	1.900
.990	1.990
.999	1.999

Notice that as Q gets closer and closer to $(1,1)$ from the left, the slope of the secant line gets closer and closer to 2 again. We will soon learn that this is written $\lim_{x \to 1^-} \frac{x^2-1}{x-1} = 2$.

Since we have approached $(1,1)$ from both the left and right and both times the value of the slope of the secant approached 2, we can say that the slope of the secant is 2. As we approach from the left or right, notice that the closer we get to $(1,1)$ the closer the secant gets to becoming a tangent line. Now, because the secant and tangent are essentially indistinguishable, we can say the slope of the tangent at $(1,1)$ is 2 and the equation of the tangent becomes $y - 1 = 2(x - 1)$.

2.1.3 Approximating the slope of a tangent from a table of values

Many times we will not be given a function in algebraic form. Instead we will be given a selection of function values in table form. Consider the following table which contains selected values of a function f.

x	0.000	0.100	0.200	0.300	0.400	0.500
$f(x)$	3.860	3.710	3.400	3.020	2.350	1.460

Again, we want to write an equation of a tangent–this time at the point $(0.200, 3.400)$. We cannot use the same procedure we used in the previous problem because we do not have the algebraic form of the function which means we cannot write a general expression for the slope of the secant. So, what will we do?

Well, we have the point, what we need is the slope... or an estimate of the slope of the tangent. We will estimate the slope of the tangent using the slope of a secant through two points close to $(0.200, 3.400)$. We have three choices:

- We can use the the point $(0.200, 3.400)$ and the point just to the left of it, $(0.100, 3.710)$ and use the slope of this secant as an estimate of the slope of the tangent. The slope of the this secant is -3.100

so our tangent line becomes $y - 3.400 = -3.100(x - 0.200)$.

- We can use the the point $(0.200, 3.400)$ and the point just to the right of it, $(0.300, 3.020)$ and use the slope of this secant as an estimate of the slope of the tangent. The slope of the this secant is -3.800 so our tangent line becomes $y - 3.400 = -3.800(x - 0.200)$.
- We can use the the points on either side of $(0.200, 3.400)$... essentially "bracketing" the point where we want the tangent. This means we use $(0.100, 3.710)$ and $(0.300, 3.020)$. The slope of the this secant is -3.450 so our tangent line becomes $y - 3.400 = -3.450(x - 0.200)$.

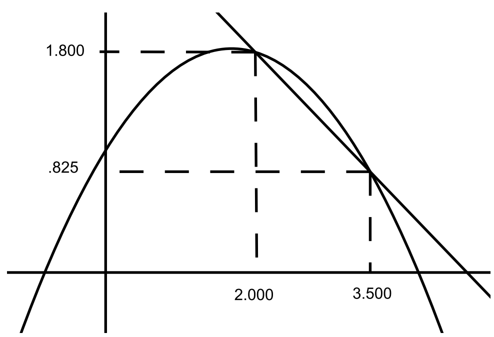
Although any of these three estimates could be used an approximation of the tangent we want but the third choice is best. Whenever you are using tables to estimate a slope, try to bracket the point in question... you'll always get a better approximation.

2.1.4 Velocity and the secant line/tangent line

Average velocity over an interval of time is given by

$$\frac{\text{change in position}}{\text{change in time}}.$$

It is a rate of change. If we consider the graph of a position function $s(t)$, the average velocity over an interval is the slope of the secant line passing through the endpoints of the interval. In the diagram below, the graph of a position function $s(t) = -0.300t^2 + t + 1$ is given. The distance units are feet and the time units are seconds. The average velocity on the interval [2.000, 3.500] can be calculated by finding the slope of the secant line through (2.000, 1.800) and (3.500, 0.825).



The slope of the secant is -0.650 and so the average velocity on [2.000, 3.500] is -.0650 feet per second.

What is more interesting is to know the velocity *at the instant when* $t = 2$ seconds. This type of velocity is called "instantaneous velocity". Instantaneous velocity and other instantaneous rates of change will be one of the core concepts of this course.

To estimate the instantaneous velocity of the object at $t = 2$, we will consider shorter and shorter time intervals until the exact velocity at $t = 2$ and the average velocity over a very tiny ("arbitrarily small" is the phrase we use) time interval become approximately the same.

Now, since we have the algebraic form of our position function, we can proceed as in the previous example when we were given an actual function. Since the average velocity is calculated by

$$\frac{\text{change in position}}{\text{change in time}}$$

we will calculate

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

which, for our function becomes

$$\frac{(-0.300t^2 + t + 1) - 1.800}{t - 2.000}.$$

where $t_1 = 2.000$ and $s(t_1) = 1.800$. The table below shows average velocities for shorter and shorter intervals of time... all starting with $t = 2.000$.

Interval	$\frac{(-0.300t^2 + t + 1) - 1.800}{t - 2.000}$
[2.000,3.500]	-0.650
[2.000,3.000]	-0.500
[2.000,2.500]	-0.350
[2.000,2.300]	-0.290
[2.000,2.100]	-0.230
[2.000,2.010]	-0.203
[2.000,2.001]	-0.200

The average velocities are getting closer and closer to -0.200 feet per second so we can say the approximate the velocity at $t = 2$ is -0.200 feet per second. Now, technically we should have approached from the other direction also... starting at something like $t_2 = 1.500$, then 1.900 and so on to see if we got the same value of -0.200. We do, trust me.

We've done quite a bit of approximating in this section. Estimation and approximation are recurring themes in this course. Using the slope of a secant to approximate the slope of a tangent is something we will do with some regularity. Keep in mind that the slope of a secant is a measure of the average rate of change in function values over an interval and the slope of a tangent is a measure of the instantaneous rate of change in function values at a point. We will, of course, move beyond estimations and will be able to calculate the slope of a tangent exactly... and thus determine instantaneous rates of change exactly. And so do with relative ease! We will also see that finding instantaneous velocity and finding the slope of a tangent to a position curve at a point are actually the same problem. In order to go beyond estimating the slope of a tangent, we will need something called the "derivative". The derivative is a limit... so that will be our next step... learning how to find the limit of a function.

2.2 The Limit of a Function

2.2.1 Introduction

We will study two basic types of limits. The first, and the topic of this section, describes the behavior of a function as the independent variable approaches a specific value. By convention, the independent variable we normally use is x, so we will refer to these types of limits as "limits as $x \to a$". The notation $x \to a$ is read "x approaches a". The second type of limit describes the behavior of a function as the independent variable increases or decreases without bound. We will refer to these limits as "limits as $x \to \pm\infty$ or "limits at infinity". In this section we will study limits as $x \to a$. The other category, limits at infinity, will come later.

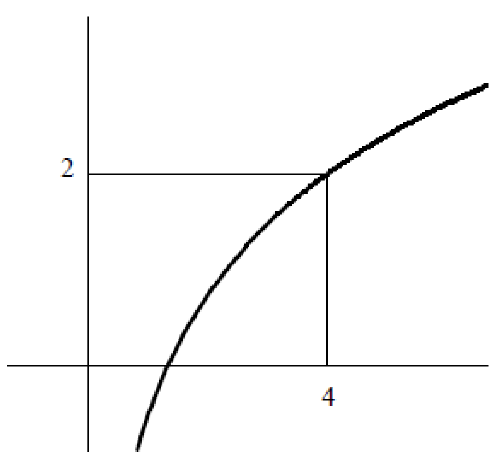
If we want to ask the question, "What happens to the function f as $x \to a$?", we use the following notation: $\lim\limits_{x\to a} f(x)$. When we approach a number, we must approach from both the left and right sides. If we are approaching a from the the right of a, the limit is called a right-hand limit. If, for example, we are approaching $x = 5$ from the right, we would find function values at 5.5, 5.2, 5.1, 5.01, 5.001 and so on. A right handed limit is denoted $\lim\limits_{x\to a^+} f(x)$. This is read "the limit of $f(x)$ as x approaches 5 from the right". Similarly, a left handed limit is denoted $\lim\limits_{x\to a^-} f(x)$ and we would evaluate the function at values like 4.5, 4.8, 4.9. 4.99, 4.999 and so on.

It is very important to remember that when we find a limit, we must approach from both sides. In fact, it is a theorem:

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$$

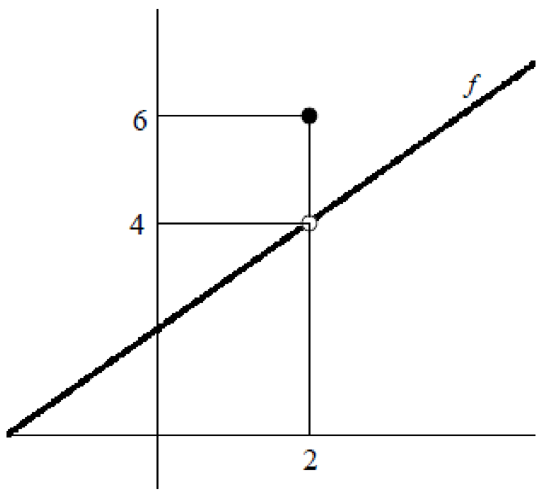
2.2.2 Finding limits from graphs

Consider the following diagram of a function f:



Note that as x gets closer and closer to 4 from the right, the value of the function approaches 2. As x gets closer and closer to 4 from the left, the value of the function also approaches 2. Since $\lim_{x\to 4^+} f(x) = 2$ and $\lim_{x\to 4^-} f(x) = 2$, we can say $\lim_{x\to 4} f(x) = 2$. *We are not saying that the function value at 4 is 2.* All we are saying is that as x *approaches* 4, the function value *approaches* 2.

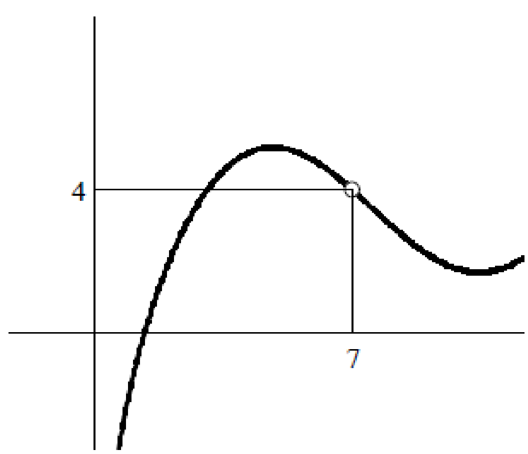
You may be asking, "Why not just find $f(4)$? After all, $f(4) = 2$." To answer this, let's look at another function. Consider the following diagram.



Notice that $f(2) = 6$ but $\lim_{x\to 2^+} f(x) = 4$ and $\lim_{x\to 2^-} f(x) = 4$ so $\lim_{x\to 2} f(x) = 4$, not 6.

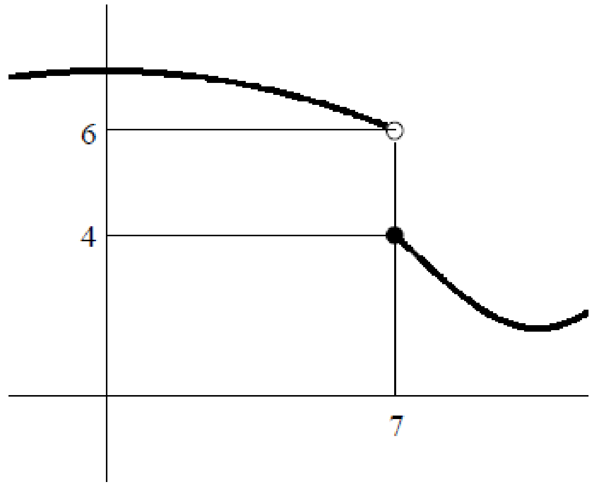
This is a critical lesson. . . finding a limit and finding a function value are not necessarily the same thing. Sometimes the limit and the function value are the same (and we like it when it is!) but this does not always happen. For now, it is vital that you understand that finding a limit and finding a function value are two *very different* things.

As a matter of fact, a function does not even have to exist at a particular x value to have a limit there! Consider the graph of the function f below:



Notice that $f(7)$ does not exist but $\lim_{x\to 7} f(x) = 4$.

Now take a look at the graph of the function below. This situation illustrates that left- and right-hand limits are not always the same.



For this piecewise function, $\lim_{x\to 7^+} f(x) = 4$ but $\lim_{x\to 7^-} f(x) = 6$ therefore, $\lim_{x\to 7} f(x)$ does not exist.

2.2.3 Finding limits when given an algebraic expression

As you will soon know, there are very simple techniques for determining the limit of a function when the function is given to us in algebraic form. For now, to find a limit as $x \longrightarrow a$ we will evaluate the expression at values of x that are increasingly close to ("arbitrarily close" is the phrase we use) but never equal to a. Remember, we must approach a from both the left and the right side.

Consider the following limit: $\lim_{x\to 1} \frac{x-1}{x^1-1}$. Notice first that the expression does not exist at $x = 1$. This is of no concern to us because a function can fail to exist at a point and still have a limit there. The tables below shows the value of $\frac{x-1}{x^2-1}$ at several values of x approaching $x = 1$ (but never actually equaling 1) from the right and left.

From the right	
x	$\frac{x-1}{x^2-1}$
1.500	0.400
1.300	0.453
1.100	0.476
1.010	0.498
1.001	0.500

From the left	
x	$\dfrac{x-1}{x^2-1}$
0.500	0.667
0.800	0.556
0.900	0.526
0.990	0.503
0.999	0.500

From the charts it appears that $\lim_{x \to 1^+} \dfrac{x-1}{x^2-1} = 0.500$ and $\lim_{x \to 1^-} \dfrac{x-1}{x^2-1} = 0.500$ and therefore we will say $\lim_{x \to 1} \dfrac{x-1}{x^2-1} = 0.500$.

It is important that you understand that we are *estimating* here. It appears from the charts that $\lim_{x \to 1} \dfrac{x-1}{x^2-1} = 0.500$ and that's the best we can do for now. Also, we are not saying that the value of $\dfrac{x-1}{x^2-1}$ at $x = 1$ is 0.500. All we are saying is that as x gets closer and closer to 1, the value of $\dfrac{x-1}{x^2-1}$ gets closer and closer to 0.500.

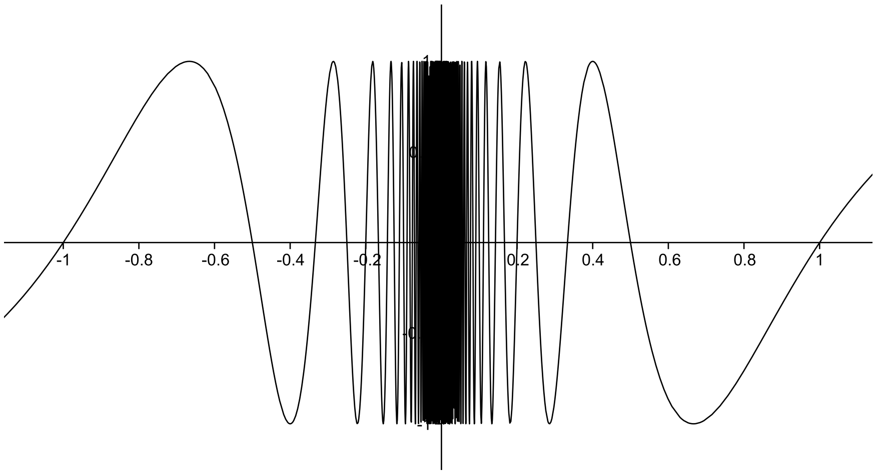
Estimating a limit using tables of values, although a useful tool, has its limitations. It just doesn't work nicely for all expressions. Let's try to estimate $\lim_{x \to 0} \sin \dfrac{\pi}{x}$ using tables of values.

From the right	
x	$\sin(\pi/x)$
0.500	0.000
0.300	0.866
0.118	1
0.100	0
0.009	-1
0.001	0.000

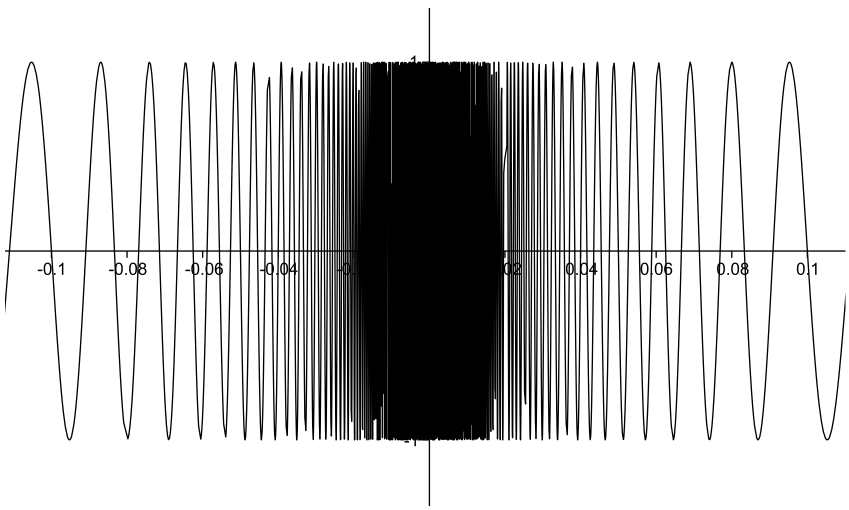
From the left	
x	$\sin(\pi/x)$
-0.500	0.000
-0.300	0.866
-0.118	-1
-0.100	0
-0.009	1
-0.001	0.000

The values of the expression do not appear to nicely approach any particular number and so our charts tell us very little. In fact, $f(x) = \sin \dfrac{\pi}{x}$ is a rather fascinating function and does not have a limit as x

approaches zero. The graph of the function on the interval $[-1, 1]$ is shown below.



Again, the graph of f, but this time on the interval $[-0.100, 0.100]$.



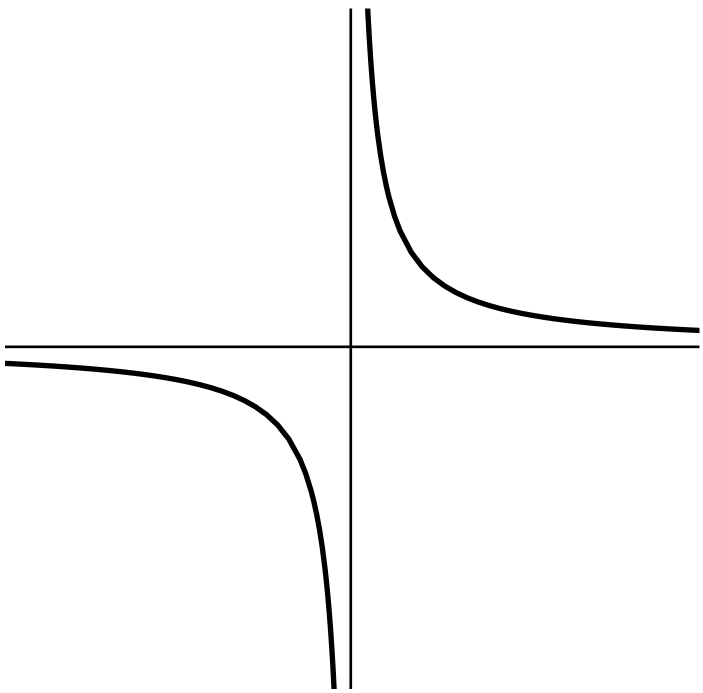
This function oscillates wildly between -1 and 1 and in fact has an infinite number of zeros between -1 and 1.

OK ... enough fun ... back to work.

We will see a veritable host of piecewise functions this year. When taking limits of a piecewise function, we need to pay close attention to where the limit is being taken and where each piece is defined. Consider $v(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$. Let's try to find $\lim_{t \to 0} v(t)$. We will need to find both the left and right hand limits. When we approach zero from the right, we will use the bottom piece and when we approach zero from the left we will use the top piece. Again, for now we are estimating limits using tables. Well, if you made a table you would find $\lim_{t \to 0^+} v(t) = 1$ but $\lim_{t \to 0^-} v(t) = 0$ so *the* limit $\lim_{t \to 0} v(t)$ does not exist. Now, if we were asked to find $\lim_{t \to 7} v(t)$, we would only need to use the bottom piece to make our tables.

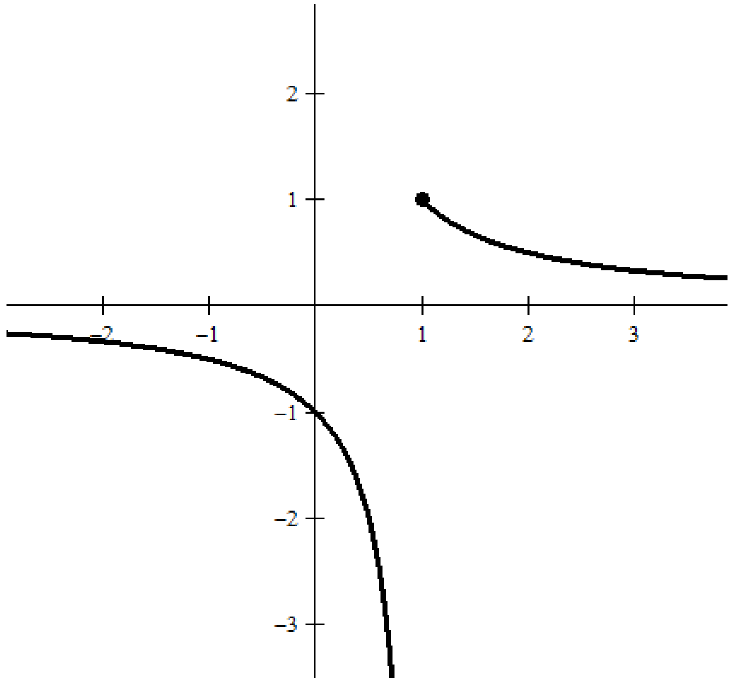
2.2.4 Non-existent limits

We've already seen several cases where a limit can fail to exist. In most of the previous examples, when a limit failed to exist, it did so because the left and right hand limits were not equal. This is not the only situation in which a limit can fail to exist. Let's take a look at $f(x) = \frac{1}{x}$ and the limit of f as $x \longrightarrow 0$. The graph of f is shown below.



As $x \longrightarrow 0^+$, the function values increase without bound. We say the function "blows up" from the right. Since the function takes on larger and larger values, it literally has no bound–no limit. We denote this by $\lim_{x \to 0^+} f(x) = \infty$. Stating $\lim_{x \to 0^+} f(x) = \infty$ is the same as saying the limit does not exist. When we say "$= \infty$" we are simply giving the reader additional information about how the limit fails to exist. If we approach zero from the left we obtain $\lim_{x \to 0^-} f(x) = -\infty$. This means that as we get closer and closer to zero from the left, the function values decrease without bound–the function "blows down" from the left. Again, the limit fails to exist. *The* limit $\lim_{x \to 0} f(x)$ then fails to exist not because the left and right hand limits were not equal but because the individual limits from each side failed to exist. Let's alter the problem just a little and consider one last example.

Consider the function $f(x) = \begin{cases} \frac{1}{x} & \text{for } x \geq 0 \\ \frac{1}{x-1} & \text{for } x < 1 \end{cases}$ and the limit $\lim_{x \to 1} f(x)$. The graph of f is shown below.



Now, $\lim_{x\to 1^+} f(x) = 1$ and $\lim_{x\to 1^-} f(x) = -\infty$. This means that the limit $\lim_{x\to 1} f(x)$ does not exist. Again, it fails to exist not because the left and right hand limits are not equal but because the left hand limit did not exist.

In summary, there are two very different ways for a limit to fail to exist. First, a limit may fail to exist because the left and right hand limits are not equal. Second, a limit may fail to exist because the function increases or decreases without bound from the left or right or both. It is the second case which will lead us to our definition of vertical asymptotes.

2.2.5 Vertical Asymptotes

You may have already noticed that whenever we have a limit that increases or decreases without bound as $x \longrightarrow a$, we have a vertical asymptote. In fact this is the definition of a vertical asymptote.

> The function f has a vertical asymptote at $x = a$ if and only if $f(a) \nexists$ and $\lim_{x\to a} f(x) = \pm\infty$.

We *must* use limits to find vertical asymptotes. A very common error that students make is to assume that just because a function fails to exist at a number, the function must have a vertical asymptote at that number. Consider the function $f(x) = \dfrac{x^2-9}{x-3}$. Now, this function clearly does not exist at $x = 3$. If you graphed the function you would see a line with a hole at the point $(3, 6)$. You would *not* see any asymptotes. There is no asymptote at $x = 3$ because f has a limit as $x \to 3$. If you approach 3 from the right or left the value of the function f gets closer and closer to 6. In other words, $\lim_{x\to 3} f(x) = 6$. If a function has a limit at a number, there will be no asymptote.

2.3 Limit Theorems

2.3.1 Introduction

We discovered in the previous section that we could answer questions about limits if we are given the graph of a function. When given an algebraic expression or a function in algebraic form, all we have been able to do thus far is estimate a limit–by using tables. In this section we will learn methods to find the exact value of a limit when presented with an algebraic expression or function. As in the previous section, we will be addressing only limits as $x \to a$, limits at infinity $(x \to \pm\infty)$ will come later.

2.3.2 How to find a limit as $x \to a$

Before we discuss the technique we actually use to find a limit as $x \to a$, we need to remind ourselves that finding a limit is *not* the same as finding a function value. That being said, the first step in finding the limit of an expression is to "plug in" the a. The next step depends on what happens as a result of "plugging in a".

There are three things that may happen when you plug in the a. First of all, you may get a constant. In this case you are finished and have found the limit.

Example 1

Find $\lim_{x\to 5} (3x + 2)$.

If we replace the x with 5 we get 17.

Since we put in the a and got out a number, we are finished and can say:

$$\lim_{x\to 5} (3x + 2) = 17$$

We need to be very clear about what we've just done. We are *not* saying that the value of $3x+2$ at $x = 5$ is 17. All we are saying is that as the x gets closer and closer to 5, the value of $3x + 2$ gets closer and closer to 17. This is an important distinction! It is so important that when you do limit problems, never make it look like you found a function value. All of the work you do "plugging in the a" needs to be done out of sight! Use scratch paper and then simply state your result. In this example, the *only* work you would show is $\lim_{x\to 5} (3x + 2) = 17$.

Example 2

Find $\lim_{x \to 3} \frac{x-3}{x+5}$.

$$\lim_{x \to 3} \frac{x-3}{x+5} = 0$$

Again all the work we did, we did out of sight. We would never want to write

$$\lim_{x \to 3} \frac{x-3}{x+5} = \frac{3-3}{5+3} = \frac{0}{8} = 0$$

because that would make it look like we're finding a function value. The problem is asking us what happens to the value of $\frac{x-3}{x+5}$ as the x gets closer and closer *but never equal to* $3 \ldots$ so never make it look like you found a function value.

Alright ... the second thing that may happen when you plug in the a is you may get the indeterminant form $0/0$. This simply means that you are not finished and must do something before trying once again to plug in the a.

If you get a $0/0$ you have several choices:

- You can factor and reduce common factors–which is allowed when we take a limit.
- You can try rationalizing the numerator.
- You can use L'Hopital's Rule ... once we learn how to use it.

So for now your options are basically factoring or rationalizing.

Example 3–Factoring

Find $\lim_{x \to 3} \frac{x^2-9}{x-3}$.

Off on our scratch paper we substitute a 3 for the x. This results in $0/0$. In this problem we will factor the expression, reduce common factors and then try plugging in the a again.

$$\begin{aligned} \lim_{x \to 3} \frac{x^2-9}{x-3} &= \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} \\ &= \lim_{x \to 3} (x+3) \\ &= 6 \end{aligned}$$

Notice that all the algebraic steps are shown, but once again there is no indication that we found a limit by finding a function value.

This might be a good time to address when we can and when we cannot reduce common factors. Consider the function

$$f(x) = \frac{x^2 - 9}{x - 3}.$$

We can rewrite f as

$$f(x) = \frac{(x+3)(x-3}{x-3}.$$

but we could not reduce the common factor and say

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x+3)(x-3)}{x-3} = x + 3.$$

We simply cannot say that $f(x) = \dfrac{x^2 - 9}{x - 3}$ and then say that $f(x) = x + 3$. These two functions have different domains and therefore cannot be *equal*.

By the same token, we could not say that the following is true

$$\frac{(x-3)(x+3)}{x-3} = x + 3$$

Two expression cannot be equivalent if we can put one number (namely $x = 3$ in this case) into one expression but not the other.

When we take a limit however, we can reduce common factors. We can say

$$\lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x \to 3} (x + 3)$$

We can do this because we are *never* letting the x be *equal* to 3! We are letting the value of x approach 3. Because we are only letting x *approach* 3, $x - 3$ will never be a zero and thus we are never dividing by zero. Cool huh?

Example 4–Rationalizing

Find $\displaystyle\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$.

$$\begin{aligned}
\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} &= \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \\
&= \lim_{x \to 9} \frac{x - 9}{(x - 9)\left(\sqrt{x} + 3\right)} \\
&= \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} \\
&= \frac{1}{6}
\end{aligned}$$

We can actually do Example 4 by factoring.

$$\begin{aligned}\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9}\frac{\sqrt{x}+3}{(\sqrt{x}-3)(\sqrt{x}+3)}\\ &= \lim_{x\to 9}\frac{1}{\sqrt{x}+3}\\ &= \frac{1}{6}\end{aligned}$$

Finally, the third thing that may happen when you plug in the a is you get an undefined expression $\ldots\frac{k}{0}$ where k is a constant. This means that you have an undefined limit. Sometimes that's all you have to say. Other times we will need to know what is happening to the function as we approach a. To do this, we need to find left and right hand limits and determine if the function increases or decreases without bound.

Example 5–Infinite limit

Find $\lim\limits_{x\to 3}\frac{1}{x-3}$

The result of substituting 3 for the x is $\frac{1}{0}$ which means the limit does not exist and the graph of the function will increase or decrease without bound as $x\to 3$. At this point we could just say

$$\lim_{x\to 3}\frac{1}{x-3}\ \nexists$$

More often, we want to tell the reader (or we need to know ourselves) how the function behaves as $x\to 3$. To do this, we will find left and right hand limits. First, think of a number just to the right of 3, like 3.001. If we substituted 3.001 into the expression, the numerator would be positive and the denominator would be positive and so the expression would be positive. This means

$$\lim_{x\to 3^+}\frac{1}{x-3}=+\infty$$

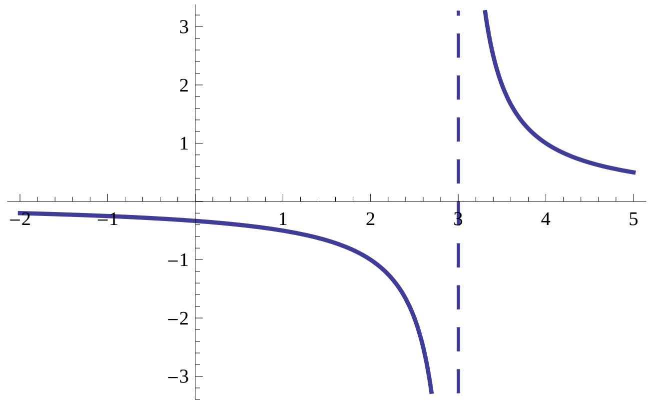
Now choose a number just to the left of 3, like 2.999. If we substitute 2.999 into the expression, the numerator is still positive but now the denominator is negative so the expression is negative. This means

$$\lim_{x\to 3^-}\frac{1}{x-3}=-\infty$$

A complete solution to this problem should look like this

$$\lim_{x\to 3}\frac{1}{x-3}\ \nexists \text{ but } \lim_{x\to 3^+}\frac{1}{x-3}=+\infty \text{ and } \lim_{x\to 3^-}\frac{1}{x-3}=-\infty$$

This means that the function $f(x)=\frac{1}{x-3}$ has a vertical asymptote at $x=3$. The graph of $f(x)=\frac{1}{x-3}$ is shown below:



Again, all the "plugging in" was done on scratch paper to avoid the appearance of finding a function value as opposed to finding a limit.

Most of the time, unless you have to factor or rationalize there is very little "work" shown when you find a limit!

2.3.3 Limits of piecewise functions

Whenever faced with a piecewise function or a function involving absolute value, we begin the problem by finding both left and right hand limits.

Example 6

Given $f(x) = \begin{cases} \sqrt{x-4} & \text{for } x > 4 \\ 8-2x & \text{for } x < 4 \end{cases}$, find $\lim_{x \to 4} f(x)$.

Note that f does not exist at $x = 4$ but this does not matter to us because a function does not have to exist at a number in order to have a limit there. To find the limit from the right, we need to use the top piece because we are approaching 4 from values of x which are slightly greater than 4. To find the limit from the left we will use the bottom piece. **The work shown below is exactly how you would present your solution!**

$$f(x) = \begin{cases} \sqrt{x-4} & \text{for } x > 4 \\ 8-2x & \text{for } x < 4 \end{cases}$$

$$\text{Since } \lim_{x \to 4^+} f(x) = 0 \text{ and } \lim_{x \to 4^-} f(x) = 0 \ \rightarrow \ \lim_{x \to 4} f(x) = 0.$$

Example 7

Given $f(x) = \begin{cases} \sqrt{x-3} & \text{for } x > 4 \\ 8-2x & \text{for } x < 4 \end{cases}$, find $\lim\limits_{x\to 4} f(x)$.

$$f(x) = \begin{cases} \sqrt{x-3} & \text{for } x > 4 \\ 8-2x & \text{for } x < 4 \end{cases} \text{, find } \lim_{x\to 4} f(x).$$

$$\text{Since } \lim_{x\to 4^+} f(x) = 1 \text{ and } \lim_{x\to 4^-} f(x) = 0 \rightarrow \lim_{x\to 4} f(x) \nexists.$$

Example 8

Given $f(x) = |x-2|$ find $\lim\limits_{x\to 2} f(x)$.

All linear absolute value functions can be written as piecewise functions and that will be our first step.

$$f(x) = \begin{cases} x-2 & \text{for } x \geq 2 \\ 2-x & \text{for } x < 2 \end{cases}$$

$$\text{Now, since } \lim_{x\to 2} f(x) = 0 \text{ and } \lim_{x\to 2} f(x) = 0 \longrightarrow \lim_{x\to 2} f(x) = 0.$$

2.3.4 Limit theorems

Most of the theorems that apply to limits are intuitive in nature and so their proofs will not be shown. We have already used many of these theorems in previous examples. They are listed here as a reference.

$$\left[\lim_{x\to a} f(x) + \lim_{x\to a} g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

The limit of a sum is the sum of the limits.

$$\left[\lim_{x\to a} f(x) - \lim_{x\to a} g(x)\right] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

The limit of a difference is the difference of the limits.

$$\left[\lim_{x\to a} f(x) \cdot g(x)\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

The limit of a product is the product of the limits.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$$

The limit of a quotient is the quotient of the limits.

$$\lim_{x\to a}cf(x)=c\lim_{x\to a}f(x)$$

Constants can be "brought outside" the limit statement.

Example: $\lim_{x\to 2}8x^3=8\lim_{x\to 2}x^3$

$$\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a}f(x)\right]^n$$

This theorem also holds for rational exponents.

Example: $\lim_{x\to -9}(7+4x)^3=\left[\lim_{x\to -9}(7+4x)\right]^3$

$\lim_{x\to a}[c]=c$ where c is a constant

Example: $\lim_{x\to 2}9=9$

$$\lim_{x\to a}f(g(x))=f\left(\lim_{x\to a}g(x)\right)$$

Example: $\lim_{x\to 0}\sin 6x=\sin\left(\lim_{x\to 0}6x\right)$

2.3.5 Limits of trigonometric functions

We will only use (for now) only two theorems involving the trigonometric functions.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

The proof of the second makes use of the first . . . and the proof of the first makes use of a theorem known as The Squeeze Theorem (sometimes called The Sandwich Theorem). The proof of The Squeeze Theorem is "beyond the scope of this course" and so we present it here without proof.

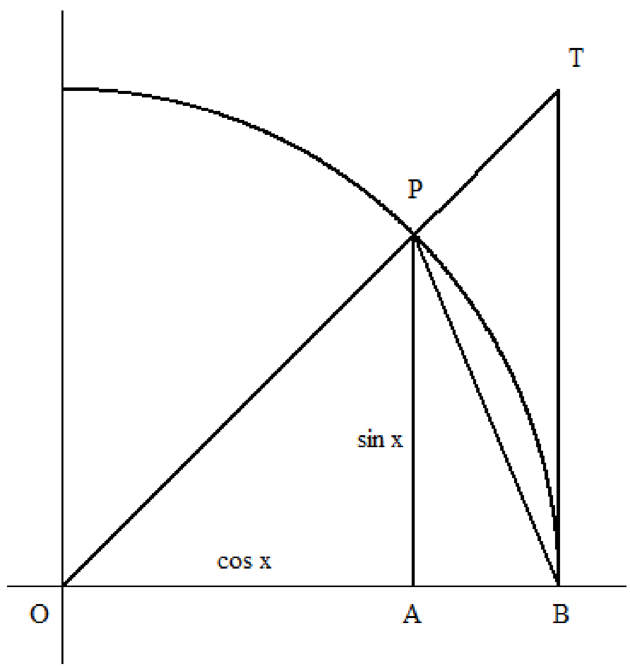
If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a
and

$\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x) = L$

The Squeeze Theorem basically tell us that if the value of a function $g(x)$ always lies between the values of the other two functions for any x, and if f and h both approach the same number as $x \to a$, then g must also approach that same function value as $x \to a$.

We will now derive the theorem that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ Consider the following diagram.



The diagram shows the portion of the unit circle in the first quadrant and thus the distance from O to A is $\cos x$ and the distance from A to P is $\sin x$.

Let's first find the coordinates of point T. We'll need them to describe several distances on the diagram. The coordinates of P are $(\cos x, \sin x)$, thus the equation of the line passing through P and the orgin is

$$y = \frac{\sin x}{\cos x}\, x.$$

Since the circle is the unit circle, the x-coordinate of T is 1. Substituting $x = 1$ into the equation of the line yields the y-coordinate of T, namely $\frac{\sin x}{\cos x}$. Therefore the coordinates of T are $\left(1, \frac{\sin x}{\cos x}\right)$.

Here's the relationship that will set up our derivation. Notice that the area of triangle BOP is less than the area of the sector of the unit circleBOP which is in turn less than the area of the triangle BOT.

The area of triangle BOP is $\frac{1}{2}\sin x$ (just one-half the base (1) times the height ($\sin x$).

The area of the sector BOP is $\frac{1}{2}x$ (one-half radius squared times the size of the angle).

The area of the triangle BOT is $\frac{1}{2}\frac{\sin x}{\cos x}$ (one-half the base times the height).

We can now write

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\frac{\sin x}{\cos x}.$$

Multiplying through by 2 yields

$$\sin x < x < \frac{\sin x}{\cos x}.$$

We now divide through by $\sin x$ (which is positive in quadrant I so the inequalities do not change.)

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

We need to now take the reciprocal of each term. As we do this we will need to change the direction of the inequalities.

$$1 > \frac{\sin x}{x} < \cos x$$

Rewriting the inequality as a "less than" statement yields

$$\cos x < \frac{\sin x}{x} < 1$$

Now, we know that $\lim_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} 1 = 1$ and so, by the Squeeze Theorem,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

The basic tactic in finding limits involving the sine function is to make parts of the expression you are working with look like the theorem. Keep in mind that as $x \to 0$, any multiple of x goes to zero. For example,

$$\lim_{x\to 0} f(x) \text{ is equivalent to } \lim_{8x\to 0} f(x).$$

Example 9

Find $\lim\limits_{x\to 0} \dfrac{\sin 7x}{2x}$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 7x}{2x} &= \lim_{x\to 0} \frac{\dfrac{\sin 7x}{7x}}{\dfrac{2x}{7x}} \\
&= \lim_{x\to 0} \frac{\sin 7x}{7x} \cdot \frac{7x}{2x} \\
&= 1 \cdot \frac{7}{2} \\
&= \frac{7}{2}
\end{aligned}$$

Note that in this example, we divided both the numerator and denominator by $7x$. We could have instead chosen to divide and then multiply the numerator by $7x$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 7x}{2x} &= \lim_{x\to 0} \frac{\dfrac{\sin 7x}{7x} \cdot 7x}{2x} \\
&= \lim_{x\to 0} \frac{7x}{2x} \\
&= \frac{7}{2}
\end{aligned}$$

In both cases, in the last steps, since the x is never going to be a zero, they can be reduced so

$$\lim_{x\to 0} \frac{7x}{2x} = \lim_{x\to 0} \frac{7}{2} = \frac{7}{2}.$$

Example 10

Find $\lim\limits_{x\to 0} \dfrac{2x}{\sin 9x}$.

$$\begin{aligned}
\lim_{x\to 0} \frac{2x}{\sin 9x} &= \lim_{x\to 0} \frac{2x}{\dfrac{\sin 9x}{9x} \cdot 9x} \\
&= \frac{2}{9}
\end{aligned}$$

Example 11

Find $\lim_{x\to 0} \frac{\sin^2 3x}{18x^2}$.

$$\begin{aligned}\lim_{x\to 0} \frac{\sin^2 3x}{18x^2} &= \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{\sin 3x}{3x} \cdot \frac{1}{2} \\ &= \frac{1}{2}\end{aligned}$$

Now that we have a limit theorem that allows us to work with functions involving the sine function, we need one for cosine.

Consider the following limit:

$$\lim_{x\to 0} \frac{1-\cos x}{x}$$

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos x}{x} &= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} \\ &= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} \\ &= \lim_{x\to 0} \frac{\sin^x}{x(1+\cos x)} \\ &= \lim_{x\to 0} \left[\frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}\right] \\ &= (1)(0) \\ &= 0\end{aligned}$$

Example 12

Find $\lim_{x\to 0} \frac{1-\cos 4x}{x}$.

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos 4x}{x} &= \lim_{x\to 0} \frac{\frac{1-\cos 4x}{4x} \cdot 4x}{x} \\ &= (0)(4) \\ &= 0\end{aligned}$$

Again, note that the common term x reduces, leaving us with $(0)(4)$.

Let's finish up with one last item. The first step you should always take when finding a limit, whether it involves trigonometric functions or not, is to substitute the a for the x and see what happens. For example, finding $\lim_{x\to 0} \sin 7x$ does not require the same procedures as the last few examples because if we substitute the zero, we just get zero out so $\lim_{x\to 0} \sin 7x = 0$. The only time you need to use the procedures we used in the previous examples is when, after substituting, you get $0/0$.

2.4 The Formal Definition of Limit

2.4.1 Introduction

The concept of limit lies at the very core of calculus. To understand calculus we must first understand what a limit is . . . what it means for a function to have a limit. Now, we would expect that such a central concept would have been well developed first, laying the foundation for all that followed. This however, was not the case. Although the invention of calculus occurred around 1690 or so, a formal, rigorous definition of limit did not exist until around 1895! For almost 200 years, mathematicians, scientists and engineers pushed ahead, extending calculus and its applications without a real foundation for its most central concept. In fact, there were mathematicians (Rolle, Berkley, etc.) who were vehemently opposed to the continued study of calculus until it could be put on rigorous foundations. The basic problem can be found in a mathematical expression you've already seen and used repeatedly the past several years $\cdots \dfrac{f(x+h)-f(x)}{h}$.

Consider the function $f(x) = x^2$.

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{h(2x+h)}{h} \end{aligned}$$

Now, as long as the h is not equal to zero we can simplify the expression to $2x + h$. Here's where the problem reared its ugly head. Mathematicians working with calculus had developed a concept called the "derivative". (We will spend a great deal of time with derivatives very soon!) Now, everyone who was doing calculus "knew" the derivative of x^2 is $2x$ and they knew how to get it. They applied the famous quotient $\frac{f(x+h)-f(x)}{h}$ to x^2. But how to get from $\frac{h(2x+h)}{h}$ to $2x$ . . . that was the problem. If the h is not zero, then you can reduce the expression to $2x+h$. However, to get from $2x+h$ to $2x$ they had to let $h = 0$! And they did! As a matter of fact, they used a somewhat infamous expression to finally rid themselves of the h. In their final step, they would say, "Suppressing the h . . .". You can see how a very un–mathematical reason like "Suppressing the h . . ." may have caused some controversy. Well, eventually Karl Wierstrauss, extending the work of Louis Cauchy, developed the modern definition of limit. This definition allows us to get around the problem with the h sometimes not being zero but then becoming zero. George Berkley (1685-1735), one of calculus' detractors, referred to the h as a "ghost of a departing quantity".

2.4.2 A verbal definition of limit

Before we move on to a more precise definition of limit, let's state the definition in more informal verbal terms.

> The limit, as x approaches a of a function f is equal to L if we can generate function values as close to L as we want by using inputs sufficiently close to, but never equal to, a.

Consider the function $f(x) = 2x + 1$. If we say that $\lim_{x \to 3} f(x) = 7$ we are saying that we can generate function values as close to 7 as we want by choosing inputs sufficiently close to but never equal to 3. It's like someone saying to us, "Hey, can you tell me how close to 3 I have to stay with my inputs if I want all my function values to be within .5 units of 7?" Of course we could! The largest function value they want is 7.5 so we could solve $2x + 1 = 7.5$ and get $x = 3.250$ which is .250 units from 3. The smallest function value they want is 6.5 so we could solve $2x + 1 = 6.5$ and get $x = 2.750$ which is also .250 units from 3. So as long as they choose x values within .250 units from 3, all their function values will be within .5 units of 7.

Now suppose the person changes their mind and now they want to stay within .1 units of 7. Could we run through our simple calculations again and tell them how close to stay to 3 ... sure we could. And exactly how long could we play this game? Forever! That means the limit statement $\lim_{x \to 3} f(x) = 7$ is true.

Time for some notation ...

The distance from L (the .5 in the previous example) is called ϵ (epsilon) and the distance from the a that we need to stay within is called δ (delta).

2.4.3 The formal definition of limit

In the formal definition (called the $\delta - \epsilon$ definition), inequalities are used to describe the distances δ and $epsilon$. Let's consider the inequality $|x - 3| < 4$. If we solve this inequality we get $x < 7$ and $x > -1$. This solution can be written $-1 < x < 7$. Notice that the midpoint of the interval $(-1, 7)$ is 3. The distance from -1 to 3 is 4 and the distance from 3 to 7 is 4. In other words, the inequality $|x - 3| < 4$ describes all the x's that are within 4 units of 3. Similarly, the inequality $|x - 5| < 2$ describes all the x's that are within 2 units of 5.

In our original problem, the person wanted all the function values to be within .500 units of 7. A very efficient way to state this is to use the inequality $|f(x) - 7| < .500$. This inequality describes all the function values within .500 units of 7–just what they wanted!

When we told them that all of the inputs must be within .250 units of 3, we could have used the inequality $|x - 3| < .250$.

Remember, epsilon (ϵ) is the distance from L within which we want all our function values. The relationship can be precisely and elegantly stated with the inequality $|f(x) - L| < \epsilon$. This inequality clearly describes all the function values that are within ϵ units of L.

Delta (δ) was the distance from a from which we needed to choose our inputs. Again, this relationship can be stated $|x - a| < \epsilon$. This inequality describes all the x's that are within δ units of a.

Now we're ready for the formal definition of limit.

Definition of Limit

$\lim_{x \to a} f(x) = L$ is true if $\forall\ \epsilon > 0\ \exists\ \delta > 0 \ni$ whenever

$$|x - a| < \delta \longrightarrow |f(x) - L| < \epsilon.$$

Note: In the definition, the "$\forall$" is the symbol for "for all" or "for any". The "$\ni$" is the symbol for "such that" and the $\longrightarrow$ means "leads to" or "implies".

The delta and the epsilon are both distances and cannot be negative. That is why the definition contains the "$\epsilon > 0$" and "$\delta > 0$". It is just to make sure everyone is playing the "game" fairly!

It is difficult to emphasize enough the importance of this definition. Its invention by Karl Weierstrauss (1815-1897) finally put Calculus on solid, rigorous foundations. Without it, the entire subject is a house of cards. It *is* the central tenant of all Calculus.

2.4.4 Working with the definition

There are two basic types of problems we face which directly involve the definition of limit. In one problem, we are given a limit statement that is true and we are given an epsilon and we need to find an appropriate delta. The second type of problem is a proof of a limit.

Example 1

Given $\lim_{x \to 3} (2x - 1) = 5$ and $\epsilon = .010$, find an appropriate δ.

For $\epsilon = .010$ we need to find an appropriate δ such that whenever $|x - 3| < \delta \longrightarrow |(2x - 1) - 5| < .010$

We proceed by working with the second inequality.

$$\begin{aligned} |(2x-1)-5| &< .010 \\ |2x-6| &< .010 \\ 2|x-3| &< .010 \\ |x-3| &< .005 \end{aligned}$$

Consider what we now have. We wanted $|x - 3| < \delta$ and we now know $|x - 3| < .005$ so we choose $\delta \leq .005$.

So what exactly have we done here? We've shown that if we stay within .005 units of 3, all the function values will be within .010 units of 5.

You may be wondering why our conclusion was "Choose $\delta \leq .005$.". We use "$\leq$" because if a delta of .005 works, then any delta smaller than .005 will work.

Example 2

Given $\lim_{x \to -2} (5x + 1) = -9$ and $\epsilon = .020$, find an appropriate δ.

For $\epsilon = .020$ we need to find an appropriate δ such that whenever $|x + 2| < \delta \longrightarrow |(5x + 1) + 9| < .020$.

$$\begin{aligned} |x+2| < \delta \;\longrightarrow\; |(5x+1)+9| &< .020 \\ |5x+10| &< .020 \\ 5|x+2| &< .020 \\ |x+2| &< .004 \end{aligned}$$

Therefore we choose $\delta \leq .004$

Thus far, all the functions we have used have been linear. In the next example we will use a non-linear function. Before we do so, we need to know one more thing about delta. By convention, we never give a delta that is bigger than one (1). It's a matter of keeping things tidy. Even if we are given a huge epsilon which algebraically generates a delta greater than one, we would choose delta to be less than or equal to one, not something larger. Remember, if a delta larger than one works, then a delta smaller would also work!

Example 3

Given $\lim\limits_{x\to 3}(x^2-2x+1)=4$ and $\epsilon = .100$, find an appropriate δ.

For $\epsilon = .100$ we need to find an appropriate δ such that whenever
$|x-3| < \delta \;\longrightarrow\; |(x^2-2x+1)-4| < .100.$

$$\begin{aligned} |x-3| < \delta \;\longrightarrow\; |x^2-2x-3| &< .100 \\ |x-3||x+1| &< .100 \\ |x-3| &< \frac{.100}{|x+1|} \end{aligned}$$

Notice that what we would normally choose for delta at this point is dependent on x. We cannot choose such a delta. We do know that the largest delta we ever give is one (1) so no matter what x we choose as an input, it must be in the interval $[2,4]$–so we proceed by considering the interval $[2,4]$.

Consider $[2,4]$.

When $x=2 \;\longrightarrow\; \dfrac{.100}{|x+1|} = \dfrac{.100}{3}.$

When $x=4 \;\longrightarrow\; \dfrac{.100}{|x+1|} = \dfrac{.100}{5}.$

Therefore choose $\delta \leq \dfrac{.100}{5}$. (Because it is the smaller of the two.

We will now move on to the second type of delta-epsilon problem, proving limit statements. We do them in much the same way we did problems in which we found appropriate deltas. The basic difference is that we now must show that there is a delta for *any* epsilon, not just a specific given epsilon.

Example 4

Prove: $\lim\limits_{x\to 6}(3x+2)=20.$

We need to show that $\forall\, \epsilon\ >\ 0\, \exists\, \delta > 0\ \ni$ whenever

$$\begin{aligned} |x-6| < \delta \ \longrightarrow\ |(3x+2)-20| &< \epsilon \\ |3x-18| &< \epsilon \\ |x-6| &< \frac{\epsilon}{3} \end{aligned}$$

Therefore choose $\delta =\ \min\ \left\{1, \frac{\epsilon}{3}\right\}$.

"choose $\delta =\ \min\ \{1, \frac{\epsilon}{3}\}$" means that, depending on what ϵ is, we always choose the smaller of 1 or $\frac{\epsilon}{3}$. If someone were to give us an epsilon of 15, we would not tell them 5, we would instead tell them to use a $\delta = 1$.

Example 5

Prove: $\lim\limits_{x\to -1} (x^2 - 5x + 1) = 7$

We need to show that $\forall\, \epsilon\ >\ 0\, \exists\, \delta > 0\ \ni$ whenever

$$\begin{aligned} |x+1| < \delta \ \longrightarrow\ |(x^2-5x+1)-7| &< \epsilon \\ |x^2-5x-6| &< \epsilon \\ |x+1||x-6| &< \epsilon \\ |x+1| &< \frac{\epsilon}{|x-6|} \end{aligned}$$

Consider $[-2, 0]$.

When $x = -2\ \longrightarrow\ \dfrac{\epsilon}{|x-6|} = \dfrac{\epsilon}{8}$.

When $x = 0\ \longrightarrow\ \dfrac{\epsilon}{|x-6|} = \dfrac{\epsilon}{6}$.

Therefore choose $\delta =\ \min\ \left\{1, \frac{\epsilon}{8}\right\}$.

2.5 Continuity

2.5.1 Introduction

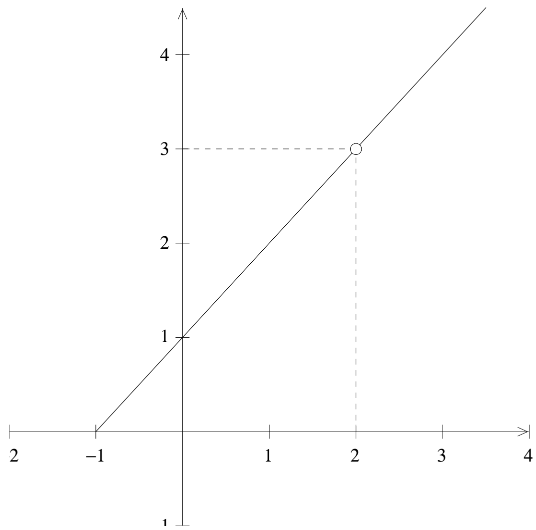
The meaning of the term *continuity* in mathematics has much that same connotation as it does in everyday usage. It implies a smoothly changing phenomenon–something that does not change abruptly or stop and then start again. Calculus was invented in part to answer questions about continuous phenomena so it makes sense that continuity is a very important concept to us. Many of the theorems we encounter will only hold for functions that are *continuous* on a particular interval. There are actually two types of continuity that concern us . . . continuity at a number and continuity on an interval. We will first deal with continuity at a number.

2.5.2 Continuity at at number

To establish the definition of continuity at a number we will examine several different functions. We will study each function at a particular number by finding a function value (if there is one) and the limit of the function at the number (if there is one). First consider the function

$$f(x) = \frac{x^2 - x - 2}{x - 2} \text{ at } x = 2.$$

The graph of f is shown below.

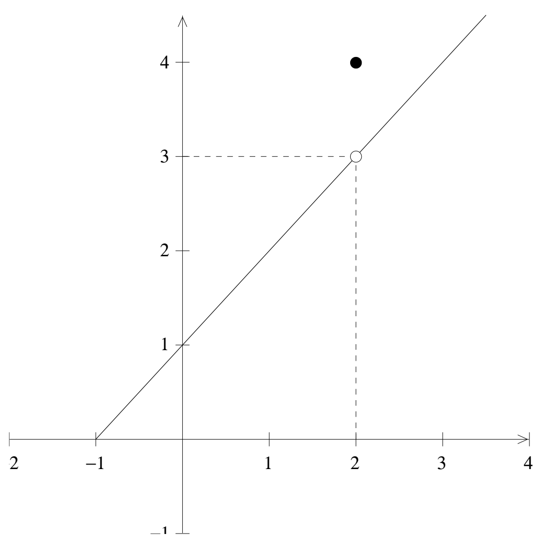


Note that $f(2)\nexists$ but $\lim_{x \to 2} f(x) = 3$. Here we have a function that has a limit but no function value at the number in question. We conclude (for reasons that we will soon know) that f is *not* continuous at $x = 2$.

Let's change the function just a little and perform the same analysis. Consider the function:

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{for } \; x \neq x \\ 4 & \text{for } \; x = 2 \end{cases} .$$

The graph of f is shown below.

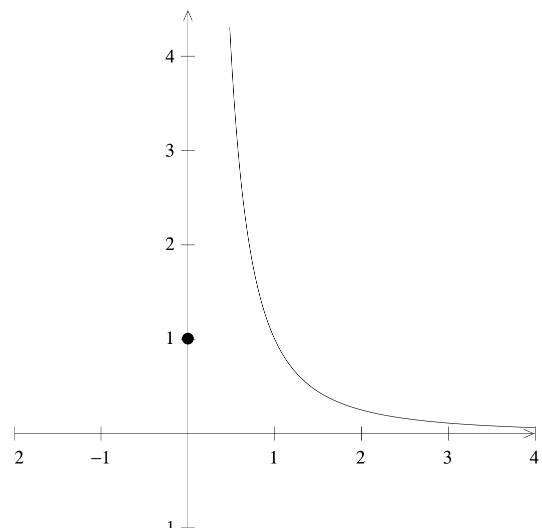


Now we can see that $f(2) = 4$ but $\lim_{x \to 2} f(x) = 3$. Even though f has both a function value and a limit at $x = 2$, we say that f is *not* continuous at $x = 2$.

Consider the function

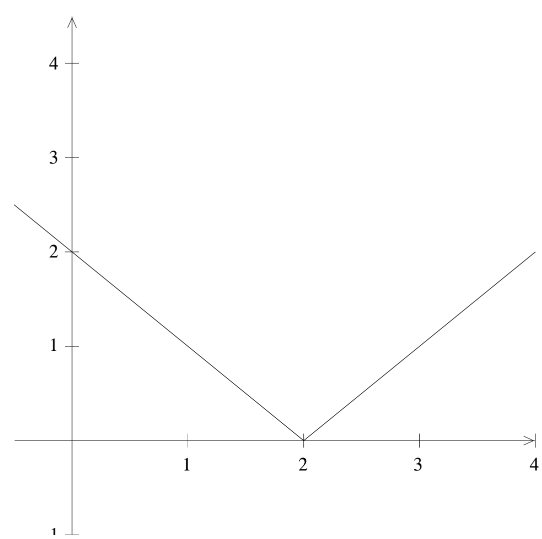
$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{for } x > 0 \\ 1 & \text{for } x = 0 \end{cases}.$$

The graph of f is given below.



Clearly, $f(0) = 1$ but $\lim_{x \to 0} f(x) \nexists$. (We cannot even approach from the left.) Unlike the previous two functions, here we have a situation where f has a function value at the number in question but has no limit there. We conclude that f is *not* continuous at $x = 0$.

Let's consider one more function, $f(x) = |x - 2|$ at $x = 2$.



For this function, $f(2) = 0$ and $\lim_{x \to 2} f(x) = 0$. The is the only function we have considered in which both the function value and the limit *exist* and are *equal*. We say that f *is* continuous at $x = 2$.

Now, it is never a good idea to generalize from a handful of examples but it would appear that for a function to be continuous at a number, the function value at the number must exist, the limit at the number must exist and they must be equal. In fact, this is the definition of continuity at a number.

Definition of Continuity at a Number

A function f is continuous at $x = a$ if and only if $f(a)$ exists, the $\lim_{x\to a} f(x)$ exists, and $f(a) = \lim_{x\to a} f(x)$

Example 1

Determine if $f(x) = \dfrac{x^2 - 25}{x - 5}$ is continuous at $x = 5$.

Continuity test at $x = 5$

$f(5) \nexists$ therefore f is not continuous at $x = 5$.

(Note that some continuity tests are very short . . . especially if the function value does not exist. If the function value does not exist, *do not* go on to find a limit–it is unnecessary!)

Example 2

Determine if $f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & \text{for } x \neq 5 \\ 7 & \text{for } x = 5 \end{cases}$ is continuous at $x = 5$.

Continuity test at $x = 5$

$f(5) = 7$ but $\lim_{x\to 5} f(x) = 10$. Since $f(5) \neq \lim_{x\to 5} f(x)$, f is not continuous at $x = 5$.

Please note a couple of common student errors. First, notice that the continuity test ends with a conclusion! Once we found the function value and and the limit we did not say "Therefore f is not continuous at $x = 5$." We made a conclusion based on our data. We said, "Since $f(5) \neq \lim_{x\to 5} f(x)$, f is not continuous at $x = 5$.". Second, our conclusion clearly stated that the function value and the limit were not equal. We did not conclude by saying "Since $f(5) = 7$ and $\lim_{x\to 5} f(x) = 10$ so f is discontinuous at $x = 5$".

Example 3

Determine if $f(x) = \begin{cases} 3x - 4 & \text{for } x \leq 2 \\ x^2 + 1 & \text{for } x > 2 \end{cases}$ is continuous at $x = 2$.

Continuity test at $x = 2$

$f(2) = 2$

$\lim_{x\to 2^+} f(x) = 5$ and $\lim_{x\to 2^-} f(x) = 2$ therefore $\lim_{x\to 2} f(x) \nexists$.

Since $\lim_{x \to 2} f(x) \nexists$, f is discontinuous at $x = 2$.

Note that three limits are shown. You must always show left, right and "the" limit on piecewise functions!

Please take note of our conclusion! We did not say that the function value and the limit are not equal! This function is discontinuous at $x = 2$ *only* because the limit failed to exist!

Example 4

Given $g(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & \text{for } x \neq 5 \\ b & \text{for } x = 5 \end{cases}$, for what value of b will g be continuous at $x = 5$?

Continuity test at $x = 5$

$$g(5) = b$$

$\lim_{x \to 5} g(x) = 10$ (We only needed to use the top piece for this limit.)

For g to be continuous at $x = 5$, $g(5)$ must be equal to $\lim_{x \to 5} g(x)$ so $b = 10$.

2.5.3 Removable vs. essential discontinuities

Not all discontinuities are created equal. Some of them can be "removed" while others are "essential" to the function and cannot be removed.

A discontinuity at $x = a$ is removable if $\lim_{x \to a} f(x)$ exists. If the limit does not exist, the discontinuity is essential.

Example 5

Determine if $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$. If discontinuous, determine if the discontinuity is removable or essential. If removable, redefine f so that it is continuous at $x = 2$.

Continuity test at $x = 2$

$f(2) \nexists$ therefore f is discontinuous at $x = 2$.

Removable or essential

Since $\lim_{x \to 2} f(x) = 4$, the limit exists and so the discontinuity is removable.

Redefine f

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{for } x \neq 2 \\ 4 & \text{for } x = 2 \end{cases}$$

As you can see, if a discontinuity is removable, you remove it by redefining f as a piecewise function in which the limit and the function value are the same.

2.5.4 Continuity on an interval

Although this is a topic that can become quite detailed, we will make use of several theorems to our work easier.

> Polynomial functions are continuous everywhere.
>
> Rational functions are continuous everywhere in their domains.
>
> Functions of the form $f(x) = \sqrt[n]{x}$ are continuous everywhere if n is odd and continuous on $[0, \infty)$ if n is even.

Example 6

Determine if $f(x) = \dfrac{3}{x-4}$ is continuous on the following intervals: $(3, 5)$, $(-1, 3)$, $[4, 9)$, $(4, 9)$.

f is not continuous on $(3, 5)$ because $f(4)$ does not exist.

f is continuous on $(-1, 3)$ because f exists for all x's in $(-1, 3)$.

f is not continuous on $[4, 9)$ because $f(4)$ does not exist.

f is continuous on $(4, 9)$ because f exists for all x's in $(4, 9)$.

2.5.5 The Intermediate Value Theorem

The Intermediate Value Theorem (IVT) belongs to a class of theorems known as "existence theorems". Existence theorems simply guarantee us that certain things will occur. We will encounter several very important existence theorems this year. The IVT is the first of these critical theorems we encounter. It is introduced now because it is based on the continuity of a function . . . which we now know about.

> **The Intermediate Value Theorem**.
>
> If f is continuous on $[a, b]$ then $\forall\ k \in (f(a), f(b))\ \exists$ a $c \in (a, b)\ \ni f(c) = k$.

Let's examine the theorem by "translating" it a more readable form. Since k is in the interval $(f(a), f(b))$, it is clearly a function value. Since c is in the interval (a, b), it is an x value. The theorem simply tells us that if a function moves smoothly from one function value to another we can find any function value between them. Consider the function $f(x) = x^2 - 3$ on $[0, 5]$. Since f is continuous on $[0, 5]$ the IVT holds. We also know that $f(0) = -3$ and $f(5) = 22$. Since f moves continuously from -3 to 22, we can generate any function value between -3 and 22. For instance, there must be an x value (the c in the theorem) such that $f(c) = 1$–because 1 is in $(-3, 22)$. To find the c we simply set the function equal to 1 and solve.

$$x^2 - 3 = 1$$
$$x^2 - 4 = 0$$
$$(x-2)(x+2) = 0$$
$$x = 2 \text{ or } x = -2$$
$$\text{Since } -2 \text{ is not in } (0,5), \text{ we choose } c = 2.$$

The IVT is used often to show that a particular function has a zero on a given interval.

Example 7

Show that $f(x) = x^3$ has at least one zero between $x = -2$ and $x = 5$ without actually finding the zero.

f is continuous on $[-2, 5]$ so the IVT holds. Since $f(-2) = -8 < 0$ and $f(5) = 125 > 0$, then by the IVT $f(x) = 0$ for some $x \in (-2, 5)$

The Intermediate Value Theorem (and other existence theorems)is also used in the process of proving other theorems. You will often see theorems being proven in which one of the steps will contain the statement "...by the Intermediate Value Theorem we know that ...".

2.6 Limits at Infinity

2.6.1 Introduction

We now turn our attention to another type of limit, limits at infinity. The basic question we will be asking is, "How does this function behave as the inputs increase or decrease without bound?" There will be quite a bit of talk about infinity in this section so let's begin by clearing up a common misconception. Infinity is not a number. It's more like a place ...a place where numbers go when they increase or decrease without bound. So when you were a little kid and argued with a friend or sibling and got to the point where you said, "Infinity plus one!" you really weren't making much sense. The term infinity us used to describe behavior–it's where we allow inputs to go and it's where some function values go.

The technique we will use is not the one normally taught in textbooks. The "official" technique involves dividing all the terms by the highest degree term and so on. Instead, we will use our common sense and our soon to be developed sense of the nature of very, very large numbers. It's something I like to call "The Big Number Game".

2.6.2 Relative size

Suppose you won one million dollars in the lottery. To show how generous you are you give one dollar to your best friend. Well, your best friend may no longer be your best friend ...after all, you only gave them one-millionth of your winnings. And yes, you're being pretty stingy. A few weeks later you win another lottery and win one trillion dollars. To make amends with your best friend you give then one million dollars. Now, they are likely much happier now–but only because they don't understand large numbers. Trouble is, you once again you have only shared one-millionth of your money!

Let's examine the expression $x^2 + 2x$. The table below shows some values of x, $2x$ and x^2.

x	x^2	$2x$	$2x$ as a percentage of x^2
10	100	20	20%
20	400	40	10%
50	2500	100	4%
100	10,000	200	2%
1,000	1,000,000	2,000	0.2%
100,000	10,000,000,000	200,000	0.002%
1,000,000	1,000,000,000,000	2,000,000	0.0002%

As x continues to get larger, $2x$ has less and less of an impact on the relative size of $x^2 + 2x$. In fact, if we let x increase without bound ($x \to \infty$) the $2x$ eventually has no real impact and can be ignored. This same line of thinking means we can ignore the x^2 in an expression like $x^3 - x^2$. As x increases or decreases without bound, x^2 loses its ability to impact the size of $x^3 - x^2$. In other words, only the term of highest degree in an expression is of any importance to us when we are taking limits at infinity.

Consider now the following limit: $\lim\limits_{x\to\infty} \dfrac{1}{x}$. If we let x increase without bound, the denominator gets larger and larger the value of the fraction gets smaller and smaller. In fact, the limit would be zero. Remember, when we say $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$ we are simply saying that as the x increases without bound, the value of the fraction gets closer and closer to zero.

Most of the problems we will face will involve fractions. To find the limit of a fraction as $x \to \pm\infty$ we begin by comparing the degree of the numerator and denominator. In the limit we just finished, the degree of the numerator was zero and the degree of the denominator was one–and that limit was zero.

Now consider $\lim\limits_{x\to\infty} \dfrac{3x-2}{x^2+5}$. Notice first that the -2 in the numerator and the $+5$ in the denominator don't matter. Only the highest degree terms matter. Also notice that the degree of the numerator is less than the degree of the denominator. In our heads (or on scratch paper) but never as part of the actual work we show, we start playing the "Big Number Game" and see that all we really have is $\dfrac{3B}{B^2}$ where B stands for a really huge number. Well, one of the B's on the top reduces with one of the B's on the bottom and we're left with $\dfrac{3}{B}$. Once again, as the x increases without bound, the value of the fraction goes to zero. So what do we actually write down? Here it is …

$$\lim_{x\to\infty} \frac{3x-2}{x^2+5} = 0$$

And that's all we write. Like many limit problems, there just isn't much work to show.

Anytime the degree of the numerator is smaller than the degree of the denominator, the limit will always go to zero as the x goes to ∞ or $-\infty$.

Now let's see what happens when the degree of the numerator is greater than the degree of the denominator. Consider the following limit:

$$\lim_{x\to\infty} \frac{x^3 - 5x^2 + x - 1}{x^2 + 6x + 8}$$

Note that the only terms of any concern are the x^3 in the numerator and the x^2 in the denominator. Now it's time for some scratch work. Playing the Big Number Game we see $\frac{B^3}{B^2}$ which reduces to B so as $x \to \infty$, the value of the fraction increases without bound in the positive direction. This means that the limit does not exist and we would say:

$$\lim_{x \to \infty} \frac{x^3 - 5x^2 + x - 1}{x^2 + 6x + 8} = \infty$$

In fact, anytime the degree of the numerator is greater than the degree of the denominator, the limit will not exist and the result will always be either ∞ or $-\infty$.

Now we will look at what happens with the degree of the numerator and denominator are the same. Consider the following limit:

$$\lim_{x \to \infty} \frac{3x - 7}{5x + 3}$$

The only terms of any concern are the $3x$ and the $5x$. The Big Number Games yields $\frac{3B}{5B}$. It won't matter how big our B gets, we can always reduce them to one. So this limit looks like this:

$$\lim_{x \to \infty} \frac{3x - 7}{5x + 3} = \frac{3}{5}$$

Whenever the degree of the numerator and denominator are the same, we carefully divide the coefficients of the highest degree terms.

The table below summarizes the technique:

Relative degree of numerator and denominator	Limit
$\circ N < \circ D$	0
$\circ N > \circ D$	$\pm\infty$
$\circ N = \circ D$	divide the coefficients of the highest degree terms

Example 1

Find $\lim_{x \to -\infty} \frac{x^2 - x}{x^3 + 5}$.

$$\lim_{x \to -\infty} \frac{x^2 - x}{x^3 + 5} = 0$$

In the case where the degree of the denominator is larger, it does not matter if x goes to positive or negative infinity, the limit is always zero.

Example 2

Find $\lim_{x \to \infty} \frac{x + 3}{\sqrt{5x2 - 7x}}$.

The degree of both the numerator and denominator is one so we carefully divide the coefficients.

$$\lim_{x\to\infty}\frac{x+3}{\sqrt{5x2-7x}}=\frac{1}{\sqrt{5}}$$

Example 3

Find $\lim\limits_{x\to-\infty}\dfrac{x+3}{\sqrt{5x2-7x}}$.

The degree of both the numerator and denominator is one so we carefully divide the coefficients.

$$\lim_{x\to-\infty}\frac{x+3}{\sqrt{5x2-7x}}=-\frac{1}{\sqrt{5}}$$

Example 3 illustrates an important point. We must very careful when the degrees are the same and the expression involves an even radical.

Example 4

Find $\lim\limits_{x\to\infty}\dfrac{5x^2-7x-1}{90x+5}$.

Since the degree of the numerator is larger, the limit will not exist and will to to positive or negative infinity.

The Big Number Game looks like this: $\dfrac{5B^2}{90B}=\dfrac{5B}{90}=\infty$.

$$\lim_{x\to\infty}\frac{5x^2-7x-1}{90x+5}=+\infty$$

Example 5

Find $\lim\limits_{x\to-\infty}\dfrac{5x^2-7x-1}{90x+5}$.

Since the degree of the numerator is larger, the limit will not exist and will to to positive or negative infinity.

The Big Number Game looks like this: $\dfrac{5(-B)^2}{90(-B)}=\dfrac{5B^2}{-90B}=\dfrac{5B}{-90}=-\infty$.

$$\lim_{x\to-\infty}\frac{5x^2-7x-1}{90x+5}=-\infty$$

In general, limits at infinity involve a lot of common sense. Just be careful with your negatives–especially when $x\to-\infty$.

2.6.3 Horizontal asymptotes

Just as we previously defined vertical asymptotes as a limit, we now define horizontal asymptotes as a limit.

> A function has a horizontal asymptote at $y = b$ if and only if
> $$\lim_{x\to\pm\infty} f(x) = b$$

You just have to be careful and make sure you do both limits... write two limit statements. Often, you will have different limits as x goes to positive and negative infinity.

Chapter 3

Derivatives

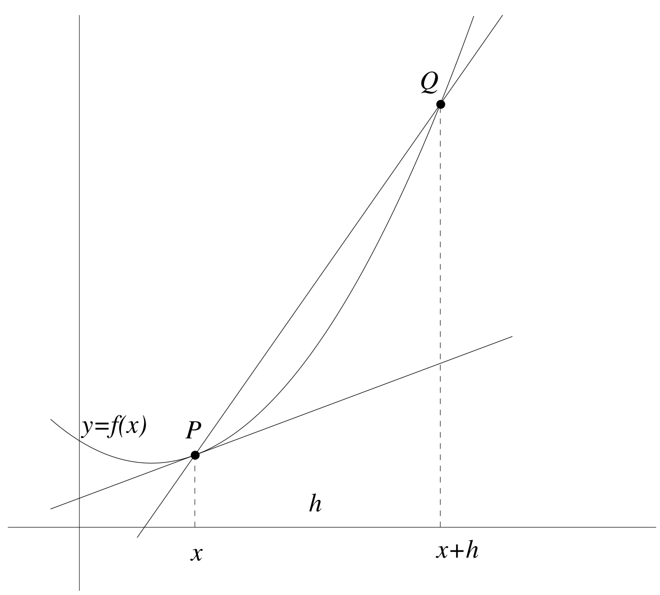
3.1 The Derivative

3.1.1 Introduction

In a previous section, we estimated the slope of a tangent line to a curve at some point P by making tables of values. We found the slope of a line through P and some other point Q. Then we choose x-coordinates for Q that got progressively closer and closer to the x-coordinate of P. In the end, we estimated the slope of the tangent by looking at what values the slope appeared to be approaching. In this section we will learn how to go beyond these estimates and actually find the exact slope of a tangent to a curve.

3.1.2 The definition of derivative

Consider the diagram below.



The slope of the secant line through P and Q is given by

$$\frac{f(x+h)-f(x)}{h}.$$

As point Q moves closer and closer to point P, the secant line and the tangent line at P become closer to being the same line. All we need is a process to let h get arbitrarily small. That process, of course, is limits! If we take the limit (shown below) of the expression as $h \to 0$, the slope of the secant line and the slope of the tangent line will be exactly the same.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

This limit is the definition of the derivative of f at x and is donoted $f'(x)$.

Definition of Derivative

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

In some instances, instead of using h to denote a small change in x, the symbol Δx is used. If we use Δx, the definition of derivative looks like this

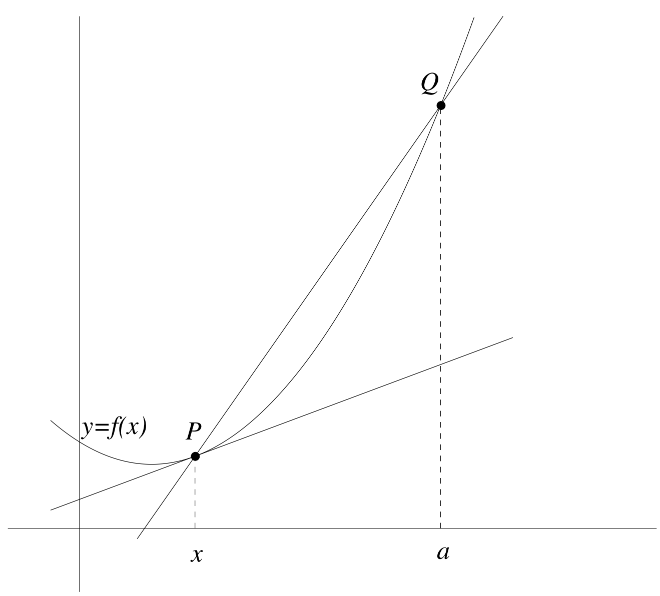
$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

The quotient we started with

$$\frac{f(x+h)-f(x)}{h}$$

is called a *difference quotient*. A difference quotient tells us the average rate of change in function values as x goes from x to $x+h$. The limit of this difference quotient as $h \to 0$ is called the derivative and tells us the instantaneous rate of change in function values at a point.

There are many ways to set up the definition of derivative. Consider the following diagram:



In this figure, instead of moving some distance h from x to get coordinates of point Q, we simply choose another x value, a. Now, the coordinates of point Q are $(a, f(a))$. The difference quotient (which is always the slope of the secant) now becomes

$$\frac{f(a)-f(x)}{a-x}.$$

In order to get the derivative, instead of letting $h \to 0$, we let $a \to x$. The definition of the derivative now becomes

$$f'(x) = \lim_{a\to x} \frac{f(a)-f(x)}{a-x}.$$

The derivative is simply a limit of a difference quotient. Listed below are several ways to express the definition of derivative.

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$f'(m) = \lim_{n\to m} \frac{f(n)-f(m)}{n-m}$$

$$f'(b) = \lim_{a\to b} \frac{f(a)-f(b)}{a-b}$$

$$f'(x) = \lim_{a\to 0} \frac{f(x+a)-f(x)}{a}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Many students get into a lazy habit of saying "…the derivative is the slope of a tangent…" or just "…the derivative is the slope…". Actually the derivative is a function or expression which can yield the slope of a tangent to a curve at any point. It can also give us other information.

How we interpret the derivative depends on the problem. In its purest sense the derivative is a function which yields the instantaneous rate of change in function values at any point. If we begin with a position function, the derivative is a function which will yield the instantaneous velocity at any point in time.

Suppose we had a function C which told us the total cost of making x number of cars. $C(100)$ would tell us the cost of making 100 cars. $C'(100)$ on the other hand would tell us the cost of the 100th car … a very different bit of information.

Consider a function $V(r)$ which describes the volume of an object with radius r. The difference quotient

$$\frac{V(r_2)-V(r_1)}{r_2-r_1}$$

tells us the average rate of change in the volume as the radius changes from r_1 to r_2. If we take a limit as $r_2 \to r_1$ we would get $V'(r_1)$, the instantaneous rate of change in the volume at r_1. The difference $r_2 - r_1$ could also be written Δr (a small change in r) and so the derivative could be expressed

$$V'(r_1) = \lim_{\Delta r\to 0} \frac{V(r_2)-V(r_1)}{r_2-r_1}.$$

Now that we have the definition of derivative, we can move on to actually finding them. In later sections, we will learn many theorems that will allow us to find derivatives quickly and easily. For now, we will use the definition of the derivative to find derivatives.

Example 1

Given $f(x) = 6x - 7$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{6(x+h) - 7 - (6x - 7)}{h} \\ &= \lim_{h \to 0} \frac{6x + 6h - 7 - 6x + 7}{h} \\ &= \lim_{h \to 0} \frac{6h}{h} \\ &= \lim_{h \to 0} 6 \\ &= 6 \end{aligned}$$

$$\therefore \; f'(x) = 6$$

Example 2

Given $f(x) = x^2 + 5x$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 + 5(x+h) - (x^2 + 5x)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 5x + 5h - x^2 - 5x}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 5h}{h} \\ &= \lim_{h \to 0} (2x + h + 5) \\ &= 2x + 5 \end{aligned}$$

$$\therefore \; f'(x) = 2x + 5$$

Example 3

Given $f(x) = \dfrac{3x+2}{2x-5}$, find $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{3(x+h)+2}{2(x+h)-5} - \dfrac{3x+2}{2x-5}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{3x+3h+2}{2x+2h-5} - \dfrac{3x+2}{2x-5}}{h} \\
&= \lim_{h\to 0} \left[\frac{1}{h}\left(\frac{(3x+3h+2)(2x-5)-(3x+2)(2x+2h-5)}{(2x+2h-5)(2x-5)}\right)\right] \\
&= \lim_{h\to 0} \left[\frac{1}{h}\left(\frac{-19h}{(2x+2h-5)(2x-5)}\right)\right] \\
&= \lim_{h\to 0} \frac{-19}{(2x+2h-5)(2x-5)} \\
&= \frac{-19}{(2x-5)(2x-5)} \\
&= \frac{-19}{(2x-5)^2}
\end{aligned}$$

$$\therefore\ f'(x) = \frac{-19}{(2x-5)^2}$$

Example 4

Given $f(x) = \sqrt{x+3}$, find $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h} \cdot \frac{\sqrt{(x+h)+3} + \sqrt{x+3}}{\sqrt{(x+h)+3} + \sqrt{x+3}} \\
&= \lim_{h\to 0} \frac{(x+h+3) - (x+3)}{h\left(\sqrt{(x+h)+3} + \sqrt{x+3}\right)} \\
&= \lim_{h\to 0} \frac{h}{h\left(\sqrt{(x+h)+3} + \sqrt{x+3}\right)} \\
&= \lim_{h\to 0} \frac{1}{h\left(\sqrt{(x+h)+3} + \sqrt{x+3}\right)} \\
&= \frac{1}{\sqrt{x+3} + \sqrt{x+3}} \\
&= \frac{1}{2\sqrt{x+3}}
\end{aligned}$$

$$\therefore\ f'(x) = \frac{1}{2\sqrt{x+3}}$$

Example 5

Write an equation of the tangent to $f(x) = x^2 + x$ at $x = 2$.

To write an equation of a line, all we need is a point and a slope.

<u>Point</u>

$$f(2) = 6 \ \rightarrow\ (2, 6)$$

<u>Slope</u>

Use the definition of derivative to obtain $f'(x) = 2x + 1$

$$f'(2) = 5 \ \therefore\ m_T = 5$$

<u>Equation of Tangent</u>

$$y - 6 = 5(x - 2)$$

3.1.3 Comments on notation

There are several ways that a derivative or the process of differentiation can be denoted. As we learn more about differentiation, we will learn more about notation.

If we are given a function "$f(x) =$", the derivative would be denoted $f'(x)$. If the function given is something like "$Q(x) =$", the derivative would be denoted $Q'(x)$.

If we are given an equation of the form "$y =$", the derivative is denoted $\frac{dy}{dx}$. For now, $\frac{dy}{dx}$ is only a symbol for the derivative, not a fraction. Later on in the course we will find out that it is, in fact, a quotient. For now, it's a symbol.

We will also see notation such as $D_x[x^2]$. This is something like a command which says, "Find the derivative of x^2." It is also the notation used when stating theorems about derivatives as in $D_x[\sin x] = cosx$. The notation "$\frac{d}{dx}$" is similar to the D_x notation.

If we need to evaluate a derivative at a specific value, say at $x = 7$, we would write $f'(7)$. If we have a derivative such as $\frac{dy}{dx} = 3x - 1$, the value at $x = 7$ would be written

$$\left.\frac{dy}{dx}\right|_{x=7} = 20.$$

3.2 Differentiation Theorems

3.2.1 Introduction

We've been using the definition of derivative to find derivatives. While this is a perfectly acceptable method, it can be quite cumbersome . . . as we already know. In this section, we will derive several theorems related to differentiation which will make finding a derivative much quicker and easier. Once we have these theorems, we will not longer have to rely on the definition of derivative to find derivatives. From now on, the only time we will have to use the definition of derivative to find a derivative is when the problem specifically states "...using the definition of derivative ...". Does this mean we are finished with the definition? No. Many of the theorems we are about to derive begin with the definition of derivative and we will return to the definition frequently throughout the course.

3.2.2 The derivative of a constant

Consider $f(x) = c$, where c is a constant.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} c - ch \\ &= 0. \end{aligned}$$

If $f(x) = c$, where c is a constant,
$$f'(x) = 0$$
or
$$D_x[c] = 0$$

Note that in the limit statement above, $c - c = 0$ but h will never be zero. The fraction $(c - c)/h$ will always have a zero in the numerator and some arbitrarily small number in the denominator–thus the value of the fraction is zero.

Example 1

Given $f(x) = 5$, find $f'(x)$.
$$f'(x) = 0.$$

3.2.3 The power rule

Before we move on to the derivation of the power rule, let's look for a particular pattern when binomials are expanded. Look at the following expansions.

$$\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{aligned}$$

Notice that the exponent on the original binomial shows up as the coefficient of the second term in the expansion. For example, the first two terms of $(a+b)^6$ are a^6+6a^5b. If we use this idea to expand $(x+h)^n$ we obtain an expression of the form
$$x^n + nx^{n-1}h + Bx^{n-2}h^2 + Cx^{n-3}h^3 + \ldots + h^n.$$

Time for the power rule. Consider the function $f(x) = x^n$ where n is any real number.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + Bx^{n-2}h^2 + Cx^{n-3}h^3 + \ldots + h^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + Bx^{n-2}h^2 + Cx^{n-3}h^3 + \ldots + h^n}{h} \\
&= \lim_{h\to 0} \frac{h[nx^{n-1} + Bx^{n-2}h + Cx^{n-3}h^2 + \ldots + h^{n-1}]}{h} \\
&= \lim_{h\to 0} [nx^{n-1} + Bx^{n-2}h + Cx^{n-3}h^2 + \ldots + h^{n-1}] \\
&= nx^{n-1}
\end{aligned}$$

During the simplification, the x^n on each end of the numerator added to zero. We then factored an h out of all the terms. That left us with nx^{n-1} followed by a series of terms all containing an h. As we took the limit $h \to 0$, all of these terms went to zeros and left us with nx^{n-1}.

The Power Rule

If $f(x) = x^n$, where n is a real number,

$$f'(x) = nx^{n-1}$$

or

$$D_x\left[x^n\right] = nx^{n-1}$$

Example 2

Given $f(x) = x^9$ and $g(x) = x^{\sqrt{7}}$, find $f'(x)$ and $g'(x)$.

$$f'(x) = 9x^8$$

$$g'(x) = \sqrt{7}x^{\sqrt{7}-1}$$

3.2.4 Three quick theorems

We will now introduce three simple but important theorems. The proofs are very straightforward and are left to the reader.

$$D_x[cf(x)] = c\, D_x[f(x)]$$

This theorem states that the derivative of a constant times a function is the constant times the derivative of the function. In other words, we just "bring along" coefficients. For example, if $f(x) = 4x^3$, the derivative is simply $4(3x^2)$ or just $12x^3$. Notice that constants that are coefficients of terms are handled differently than constants that stand alone.

$$D_x\left[f(x) + g(x)\right] = f'(x) + g'(x)$$

$$D_x\left[f(x) - g(x)\right] = f'(x) - g'(x)$$

These two theorems state that the derivative of a sum is the sum of the derivatives and the derivative of a difference is the difference of the derivatives. The next example makes use of all three of these theorems.

Example 3

Given $f(x) = 5x^4 - 3x^2 + 8$, find $f'(x)$.

$$f'(x) = 20x^3 - 6x$$

Note that the derivative of the constant on the "end" is zero.

Example 4

Given $f(x) = \sqrt{x} + \sqrt[5]{x^2}$, find $f'(x)$.

$$\begin{aligned} f(x) &= x^{1/2} + x^{2/5} \\ f'(x) &= \frac{1}{2}x^{-1/2} + \frac{2}{5}x^{-3/5} \\ &= \frac{1}{2\sqrt{x}} + \frac{2}{5\sqrt[5]{x^3}} \end{aligned}$$

3.2.5 The product rule

We already know that the derivative of a sum is the sum of the derivatives, but is the derivative of a product simply the product of the derivatives? If this were true, then the derivative of $f(x) = (x^3 + x)(5x^2 - 2x)$ would be $f'(x) = (3x^2 + 1)(10x - 2)$…but it's not. Unfortunately, it's not that easy. To differentiate a product we use something called the "product rule".

Let $p(x) = f(x)g(x)$. Using the definition of derivative we can write

$$p'(x) = \lim_{h \to 0} \frac{p(x+h) - p(x)}{h}.$$

Now, because $p(x) = f(x)g(x)$,

$$p'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

In order to move forward, we are going to subtract, and then add, the quantity $f(x+h)g(x)$ to the numerator. Actually we are going to add then subtract it in the middle of the numerator. We're just adding zero. It is a common tactic in mathematics in proofs and derivations.

$$p'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}.$$

We now separate the quotient into two quotients.

$$p'(x) = \lim_{h \to 0} \left[\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h} \right]$$

In the first fraction, $f(x+h)$ is a common term in the numerator and can be factored out. In the second fraction, $g(x)$ is a common term and can be factored out.

$$p'(x) = \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}\right]$$

Inside the brackets we now have four pieces and we will take the limit of each as $h \to 0$.

$$\lim_{h\to 0} f(x+h) = f(x)$$

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = g'(x) \text{ This limit is the definition of the derivative of } g.$$

$$\lim_{h\to 0} g(x) = g(x) \text{ This is true because } g(x) \text{ has nothing to do with } h!.$$

$$\lim_{h\to 0}\frac{f(x+h-f(x)}{h} = f'(x) \text{ This limit statement is the definition of the derivative of } f.$$

Let's put them all together now. We'll begin with our last step in the derivation.

$$p'(x) = \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}\right]$$

$$p'(x) = f(x)g'(x) + g(x)f'(x)$$

The Product Rule

If $h(x) = f(x)g(x)$

then

$$h'(x) = f(x)g'(x) + g(x)f'(x)$$

or

$$D_x[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In words you will never forget, the product rule says: **The derivative of a product is . . . the first times the derivative of the second plus the second times the derivative of the first**.

Note: Some students like to use the commutative property of addition to write the product rule as $D_x[f(x)g(x)] = f'(x)g(x)+g'(x)f(x)$. I suppose this is because some students are so anxious to take a derivative that they want to start right off with $f'(x)$. I highly discourage this! It has been my experience that students who memorize the product rule "backwards" will more often than not memorize the next theorem (the quotient rule) incorrectly. I'm not sure why this happens, but it happens. Trust me. Please, when you recite the product rule, always begin with "the first times the derivative of the second . . . "

We haven't yet run into any functions where the product rule is absolutely necessary. This will change once we learn the derivatives of the trigonometric functions and the chain rule.

Example 5

Given $h(x) = x^3 f(x)$, find $h'(x)$.

$$h'(x) = x^3 f'(x) + f(x) \cdot 3x^2$$

3.2.6 The quotient rule

The derivative of a product is not the product of the derivatives (we need the product rule) so it follows that the derivative of a quotient is not the derivative of the numerator divided by the derivative of the denominator.

The typical derivation of the quotient rule is very similar to the derivation of the product rule. There is an alternate route to the quotient rule ... and that's what we'll do now.

The general rule of the game in mathematics is that when trying to prove or derive a formula, you can use any theorem you have already proven or derived. Since we have a product rule, we'll use it to get at the quotient rule.

Consider the function $h(x) = \dfrac{f(x)}{g(x)}$.

$$f(x) = h(x)g(x)$$

$$f'(x) = h(x)g'(x) + g(x)h'(x)$$

Now, solve for $h'(x)$.

$$h'(x) = \frac{f'(x) - h(x)g'(x)}{g(x)}$$

We now need to replace the $h(x)$ with $\dfrac{f(x)}{g(x)}$.

$$h'(x) = \frac{f'(x) - \dfrac{f(x)}{g(x)}g'(x)}{g(x)}$$

Now multiply the numerator and denominator by $g(x)$.

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

The Quotient Rule

$$\text{If } h(x) = \frac{f(x)}{g(x)}$$

then

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

or

$$D_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Again, in words you will never forget, the quotient rule says: **The derivative of a quotient is ...the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator over the denominator squared**.

It is, unfortunately, a very common error to reverse the terms in the numerator of the quotient rule. Just remember, it's the presence of a denominator that makes a quotient a quotient ...it's the critical part of a quotient so always start your quotient rule with **the denominator**.

Example 6

Given $f(x) = \dfrac{x-4}{8x+1}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{(8x+1)(1) - (x-4)(8)}{(8x+1)^2} \\ &= \frac{8x+1-8x+32}{(8x+1)^2} \\ &= \frac{33}{(8x+1)^2} \end{aligned}$$

Example 7

Given $f(x) = \dfrac{10}{x^4}$, find $f'(x)$.

We will do this problem two ways. First using the quotient rule and then rewriting the function to avoid the quotient rule.

$$\begin{aligned} f'(x) &= \frac{(x^4)(0) - (10)(4x^3)}{x^8} \\ &= \frac{-40x^3}{x^8} \\ &= -\frac{40}{x^5} \end{aligned}$$

Now we rewrite f.

$$\begin{aligned} f(x) &= 10x^{-4} \\ f'(x) &= -40x^{-5} \\ &= -\frac{40}{x^5} \end{aligned}$$

It is clearly easier to differentiate after rewriting the function with negative exponents. You should never avoid the quotient rule but when your numerator is a constant, it is easier to bring up the denominator using negative exponents.

Example 8

Given $f(x) = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{(\sqrt{x}+1)\left(\dfrac{1}{2\sqrt{x}}\right) - (\sqrt{x}-1)\left(\dfrac{1}{2\sqrt{x}}\right)}{(\sqrt{x}+1)^2} \\ &= \frac{\sqrt{x}+1-\sqrt{x}+1}{2\sqrt{x}\,(\sqrt{x}+1)^2} \\ &= \frac{2}{2\sqrt{x}\,(\sqrt{x}+1)^2} \\ &= \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2} \end{aligned}$$

Example 9

Given $f(x) = \dfrac{3x-7}{x^3-6x+8}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{(x^3-6x+8)(3)-(3x-7)(3x^2-6)}{(x^3-6x+8)^2} \\ &= \frac{-6x^3+21x^2-18}{(x^3-6x+8)^2} \end{aligned}$$

3.3 Derivatives of the Trigonometric Functions

3.3.1 Introduction

We're going to get right to it in this section. Now that we have the definition of derivative, the power rule, the product rule, the quotient rule and several other differentiation theorems, we're ready to find the derivatives of the six trigonometric functions. In order to avoid confusion with the hyperbolic trigonometric functions ($\sinh x$, $\cosh x$, etc.) we will use Δx in the definition of derivative.

3.3.2 Derivative of the sine function

Let $f(x) = \sin x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[\sin x \cos \Delta x + \cos x \sin \Delta x] - \sin x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x - \sin x + \cos x \sin \Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[\frac{\sin x \cos \Delta x - \sin x}{\Delta x} + \frac{\cos x \sin \Delta x}{\Delta x} \right] \\ &= \lim_{\Delta x \to 0} \left[\sin x \frac{\cos \Delta x - 1}{\Delta x} + \cos x \frac{\sin \Delta x}{\Delta x} \right] \end{aligned}$$

From our previous work with trigonometric limits we know that

$$\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} = 0 \text{ and } \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1.$$

$$\begin{aligned} f'(x) &= (\sin x)(0) + (\cos x)(1) \\ &= \cos x \end{aligned}$$

If $f(x) = \sin x \;\rightarrow\; f'(x) = \cos x$
or
$D_x[\sin x] = \cos x$

3.3.3 Derivative of the cosine function

Let $f(x) = \cos x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - -\cos x - \sin x \sin \Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[\cos x \frac{\cos \Delta x - 1}{\Delta x}\right] - \lim_{\Delta x \to 0} \left[\sin x \frac{\sin \Delta x}{\Delta x}\right] \\ &= (\cos x)(0) - (\sin x)(1) \\ &= -\sin x \end{aligned}$$

If $f(x) = \cos x \;\rightarrow\; f'(x) = -\sin x$
or
$D_x[\cos x] = -\sin x$

3.3.4 Derivative of the tangent function

In order to derive the remaining trigonometric derivatives, we can use our quotient rule and avoid having to use the definition of derivative.

Let $f(x) = \tan x$.

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$\begin{aligned} f'(x) &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

If $f(x) = \tan x \;\rightarrow\; f'(x) = \sec^2 x$
or
$D_x[\tan x] = \sec^2 x$

3.3.5 Derivative of the secant function

Let $f(x) = \sec x$.

$$f(x) = \sec x = \frac{1}{\cos x}$$

$$\begin{aligned} f'(x) &= \frac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x} \\ &= \frac{\sin x}{\cos^2 x} \\ &= \frac{1}{\cos x}\,\frac{\sin x}{\cos x} \\ &= \sec x\;\tan x \end{aligned}$$

If $f(x) = \sec x \;\rightarrow\; f'(x) = \sec x\;\tan x$
or
$D_x[\sec x] = \sec x\;\tan x$

3.3.6 Derivative of the cotangent function

Let $f(x) = \cot x$.

$$f(x) = \tan x = \frac{\cos x}{\sin x}$$

$$\begin{aligned} f'(x) &= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} \\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \qquad = -\frac{1}{\sin^2 x} \\ &= -\csc^2 x \end{aligned}$$

If $f(x) = \cot x \;\rightarrow\; f'(x) = -\csc^2 x$
or
$D_x[\cot x] = -\csc^2 x$

3.3.7 Derivative of the cosecant function

Let $f(x) = \csc x$.

$$f(x) = \csc x = \frac{1}{\sin x}$$

$$\begin{aligned} f'(x) &= \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x} \\ &= -\frac{\cos x}{\sin^2 x} \\ &= -\frac{1}{\sin x}\,\frac{\cos x}{\sin x} \\ &= -\csc x\ \cot x \end{aligned}$$

If $f(x) = \csc x \;\rightarrow\; f'(x) = -\csc x\ \cot x$
or
$D_x[\csc x] = -\csc x\ \cot x$

Example 1

Given $f(x) = 3\sin x$, find $f'(x)$.

$$f'(x) = 3\cos x$$

Example 2

Given $f(x) = x^2 \sin x$, find $f'(x)$.

$$\begin{aligned} f'(x) &= (x^2)(\cos x) + (\sin)(2x) \\ &= x^2 \cos x + 2x \sin x \end{aligned}$$

Example 3

Given $f(x) = \dfrac{2\cos x}{x+1}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{(x+1)(-2\sin x) - (2\cos x)(1)}{(x+1)^2} \\ &= \frac{-2(x+1)\sin x - 2\cos x}{(x+1)^2} \end{aligned}$$

Example 4

Given $f(t) = \sin t \tan t$, find $f'(t)$.

$$\begin{aligned} f'(x) &= (\sin t)(\sec^2 t) + (\tan t)(\cos t) \\ &= (\sin t)(\sec^2 t) + \frac{\sin t}{\cos t}\cos t \\ &= (\sin t)(\sec^2 t) + \sin t \\ &= \sin t(\sec^2 t + 1) \end{aligned}$$

We should make note of the wonderful symmetry in the derivatives of the trigonometric functions. Sine becomes cosine, cosine becomes negative sine, etc. It's another reason the trigonometric functions are so important in mathematics and why mathematicians love them so much. The trigonometric functions are extremely well-behaved.

Notice that in all our current derivative theorems for the trigonometric functions, the argument of the trigonometric function is a single variable. Although we can now find the derivative of $\sin x$, we cannot yet find the derivative of $\sin 5x$ or $\cos x^2$ or $\tan(3x^4 - 6x)$. Functions and expressions such as these are compositions and will require something called "the chain rule" ... which is next up on our agenda!

3.4 The Chain Rule

3.4.1 Introduction

Although we have discussed quite a few theorems used for finding derivatives, we have confined ourselves to fairly simple functions and expressions. Right now we can differentiate $f(x) = x^7$, but not $f(x) = (4x^3 + 3x)^7$... at least without expanding the binomial (good luck!). We can find the derivative of $g(x) = \sin x$ but not $g(x) = \sqrt{\sin x}$ or $h(x) = \sec^8(5x - 2)$. Until now, we have never dealt with the composition of two or more functions. To differentiate compositions, we need the chain rule.

We use the product rule when we have a product, we use the quotient rule when we have a quotient. We will *always* use the chain rule... we *always* chain. In fact, you've been chaining all along–you just didn't know it!

3.4.2 The chain rule

In order to derive the chain rule, we will use a different definition of derivative than we have previously used when we derived derivative theorems. We will use

$$f'(x) = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x}.$$

Let $h(x) = f(g(x))$.

$$\begin{aligned} h'(x) &= \lim_{x_1 \to x} \frac{h(x_1) - h(x)}{x_1 - x} \\ &= \lim_{x_1 \to x} \frac{f(g(x_1) - f(g(x))}{x_1 - x} \end{aligned}$$

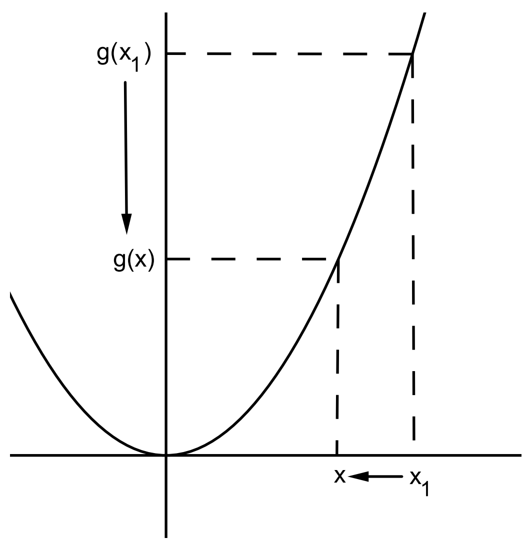
In deriving the product and quotient rule we used a technique in which we added zero to change the way an expression looked…and were thereby about to move on with the derivation. Here we use another common technique to change the appearance of an expression, we multiply by one.

$$h'(x) = \lim_{x_1 \to x} \frac{f(g(x_1) - f(g(x))}{x_1 - x} \cdot \frac{g(x_1) - g(x)}{g(x_1) - g(x)}$$

The commutative property of multiplication allows us to commute the denominators.

$$h'(x) = \lim_{x_1 \to x} \frac{f(g(x_1) - f(g(x))}{g(x_1) - g(x)} \cdot \frac{g(x_1) - g(x)}{x_1) - x}$$

Let's consider the first limit (the first quotient). If g is a well-behaved function, as x_1 approaches x, $g(x_1)$ will approach $g(x)$. The diagram below illustrates this fact.



This fact allows us to rewrite the limit of the first quotient as

$$\lim_{g(x_1) \to g(x)} \frac{f(g(x_1)) - f(g(x))}{g(x_1) - g(x)}.$$

This limit, if you look closely enough is the definition of $f'(g(x))$! The second limit

$$\lim_{x_1 \to x} \frac{g(x_1) - g(x)}{x_1 - x}$$

is the definition of $g'(x)$.

Putting it all together results in

$$\begin{aligned} h'(x) &= \lim_{x_1 \to x} \frac{f(g(x_1) - f(g(x))}{g(x_1) - g(x)} \cdot \frac{g(x_1) - g(x)}{x_1) - x} \\ &= f'(g(x)) \cdot g'(x) \end{aligned}$$

If $h(x) = f(g(x)) \to h'(x) = f'(g(x)) \cdot g'(x)$
or
$D_x[f(g(x))] = f'(g(x)) \cdot g'(x)$

The best way to learn how to use the chain rule is to see and do actual problems.

Consider $h(x) = (7x - 2)^{45}$. Now, in theory we could just expand the binomial and then use our power rule ... see ya next year! Instead, look at $h(x)$ as the composition of two functions, $g(x) = 7x - 2$ and $f(x) = x^{45}$. g is the "inside" function and f is the "outside" function. Now, $f'(x) = 45x^{44}$ so $f'(g(x)) = 45(7x - 2)^{44}$. $g'(x) = 7$. Multiplying both together we obtain $h'(x) = 45(7x - 2)^{44}(7)$ or $h'(x) = 315(7x - 2)^{44}$.

Another way to look at a function like $f(x) = (7x-2)^{45}$ is to "see" an outside and inside function. In this case the outside function is $something^{45}$ and the inside function is $7x - 2$. To find the derivative think "the derivative of $something^{45}$ is $45something^{44}$ times the derivative of the $something$. This would yield $44(7x - 2)^{44}(7)$.

When you look at a problem, you should try to "see" the outside function. Don't worry, the more problems you do, the better you'll get at "chaining".

Consider $f(x) = (x^2 - 6x)^{20}$.

- What you should see: $something$ to the 20th
- What you should say to yourself: 20 times the $something$ to the 19th times the derivative of the $something$
- The derivative: $f'(x) = 20(x^2 - 6x)^{19}(2x - 6)$

Consider $f(x) = \cos^5 x$.

- What you should see: $something$ to the 5th
- What you should say to yourself: 5 times the $something$ to the 4th times the derivative of the $something$
- The derivative: $f'(x) = (5\cos^4 x)(-\sin x) = -5\cos^4 x \sin x$

Consider $f(x) = \sin(5x - 4)$.

- What you should see: the sine of $something$
- What you should say to yourself: cosine of the $something$ times the derivative of the $something$

- The derivative: $f'(x) = (\cos(5x-4))\,(5) = 5\cos(5x-4)$

Special note about $\sqrt{x}$ and $\sqrt{f(x)}$: We have to take the derivative of the square root of functions constantly. It's worth your while to memorize the format of the derivative. The derivative of $\sqrt{x}$ is always

$$\frac{1}{2}x^{-1/2}$$

which simplifies to

$$\frac{1}{2\sqrt{x}}.$$

We will use this fact to generate a theorem for the derivative of $\sqrt{f(x)}$.

Consider $g(x) = \sqrt{f(x)}$.
We could look at this as $something$ to the 1/2. The the derivative would become

$$\frac{1}{2}\sqrt{f(x)} \text{ times the derivative of } f(x) \text{ which is } f'(x).$$

So

$$g'(x) = \frac{1}{2}\sqrt{f(x)}f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}.$$

This is a theorem you should always remember because it makes your life easier…

$$D_x\left[\sqrt{f(x)}\right] = \frac{f'(x)}{2\sqrt{f(x)}}$$

Using this theorem makes taking the derivative of something like $f(x) = \sqrt{x^5 - 3x^4 + 6}$ very easy.

$$f'(x) = \frac{5x^4 - 12x^3}{2\sqrt{x^5 - 3x^4 + 6}}$$

3.4.3 Changes in differentiation theorems

Now that we have the chain rule, many of our differentiation theorems will change to reflect the fact that they work for compositions. We will not change the way the product or quotient rule look–you just have to remember to chain.

The Power Rule

$$D_x\left[f(x)^n\right] = nf(x)^{n-1}f'(x)$$

Trigonometric Derivatives

$$D_x[\sin f(x)] = \cos f(x) \cdot f'(x)$$
$$D_x[\cos f(x)] = -\sin f(x) \cdot f'(x)$$
$$D_x[\tan f(x)] = \sec^2 f(x) \cdot f'(x)$$
$$D_x[\cot f(x)] = -\csc^2 f(x) \cdot f'(x)$$
$$D_x[\sec f(x)] = \sec f(x) \tan f(x) \cdot f'(x)$$
$$D_x[\csc f(x)] = -\csc f(x) \cot f(x) \cdot f'(x)$$

For the trigonometric derivatives, the $f'(x)$ is actually put out front more often than not. This avoids "argument confusion". For example, if

$$f(x) = \sin 7x,$$

the derivative is written as

$$f(x) = 7\cos 7x,$$

instead of $\cos 7x \cdot 7$ which looks like $\cos 49x$ which is not correct.

3.4.4 Another form of the chain rule

The chain rule can also be expressed in the following form:

The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This form is used to differentiate when the composition is actually given as two separate equations.

Consider $y = \sqrt{x^3 + 5x}$. For this equation, we could consider $y = \sqrt{u}$ and $u = x^3 + 5x$. Using the chain rule we obtain

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad \frac{du}{dx} = 3x^2 + 5.$$

Multiplying we get the derivative with respect to x.

$$\frac{dy}{dx} = \frac{3x^2 + 5}{2\sqrt{u}}$$

Now, substituting the $u = x^3 + 5x$ yields

$$\frac{dy}{dx} = \frac{3x^2 + 5}{2\sqrt{x^3 + 5x}}$$

3.4.5 Chain rule examples

Example 1

Given $f(x) = \sqrt{x^2+1}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{2x}{2\sqrt{x^2+1}} \\ &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

Example 2

Given $g(x) = \sin x^3$, find $g'(x)$.

$$\begin{aligned} g'(x) &= (\cos x^3)(3x^2) \\ &= 3x^2 \cos x^3 \end{aligned}$$

(In this problem you should see sine of *something*.)

Example 3

Given $g(x) = \sin^3 x$, find $g'(x)$.

$$\begin{aligned} g'(x) &= 3\sin^2 x \cos x \\ &= 3x^2 \cos x^3 \end{aligned}$$

(In this problem you should see *something* cubed.)

Example 4

Given $h(x) = \sec x^5$, find $h'(x)$.

$$\begin{aligned} h'(x) &= (\sec x^5 \tan x^5)(5x^4) \\ &= 5x^4 \sec x^5 \tan x^5 \end{aligned}$$

(In this problem you should see the secant of*something*.)

Example 5

Given $f(x) = \cos^5(3x+5)$, find $f'(x)$.

In this problem we have the composition of three functions: *something* to the 5th, the cosine of *something* and $3x+5$. This means that the derivative will consist of 3 parts.

$$\begin{aligned} f'(x) &= \left[5\cos^4(3x+5)\right][-\sin(3x+5)](5) \\ &= -15\cos^4(3x+5)\sin(3x+5) \end{aligned}$$

Example 6

Given $f(x) = \dfrac{5}{(x^3 - 8x^2 + 3)^6}$, find $f'(x)$.

Here's something to remember ... I never avoid the quotient rule but if my numerator is just a number, I will always bring up the denominator with a negative exponent! It's one of the advantages of having the chain rule.

$$\begin{aligned} f'(x) &= 5(x^3 - 8x^2 + 3)^{-6} \\ &= -30(x^3 - 8x^2 + 3)^{-7}(3x^2 - 16x) \\ &= \frac{-30(3x^2 - 16x)}{(x^3 - 8x^2 + 3)^7} \end{aligned}$$

Example 7

Given $f(x) = \cos^2(\cos x)$, find $f'(x)$.

$$f'(x) = [2\cos(\cos x)]\,[-\sin(\cos x)][-\sin x]$$

Example 8

Given $h(x) = f(g(x))$ and $f'(7) = 4$ and $g'(2) = 5$ and g(2)=7, find $h'(2)$.

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ h'(2) &= f'(g(2))g'(2) \\ &= f'(7)(5) \\ &= (4)(5) \\ &= 20 \end{aligned}$$

Example 9

Given $p(x) = h(f(g(x)))$, find $p'(x)$.

$$p'(x) = h'(f(g(x)) \cdot f'(g(x)) \cdot g'(x)$$

3.4.6 The three types of tangent to a curve problems

Throughout this course we will be writing equations of tangents and normals to curves. Remember, a normal is perpendicular to the tangent. These problems fall into three categories which we will refer to as Category I, Category II and Category III. These are not official names, just names we will use to distinguish the different types.

- Category I: given and curve and a point (or an x-coordinate) on the curve

- Category II: given a curve and a line to which the tangent must be parallel or perpendicular
- Category III: given a curve and a point not on the curve through which the tangent(s) must pass

Category I strategy
The point will be given. If you are given only an x-coordinate, $x = a$, find $f(a)$ to get the y-coordinate. Then find $f'(a)$. Now you have a point and the slope–use the point-slope form for a line to write the equation of the tangent.

Category I example

Write an equation of a tangent to $y = x^3 + 1$ at $x = 3$.

Point

When $x = 3 \ \rightarrow \ y = 28 \ \therefore \ (3, 28)$

Slope

$$\frac{dy}{dx} = 3x^2 \ \rightarrow \ \left.\frac{dy}{dx}\right|_{x=3} = 27 \ \rightarrow \ m_T = 27$$

Equation of tangent

$$y - 28 = 27(x - 3)$$

Category II strategy
To find the slope of the tangent, find the slope of the given line. Then find the slope of the tangent using the derivative. Now set these two equal to each other ...this will give you an x value. Plug this into the original function to the the y-coordinate. You now have a point and a slope so you can write the equation of the tangent.

Catgegory II example

Write an equation of a tangent to $y = x^3 + 1$ that is parallel to $3x - 4y = 7$.

Slope from given line

$$m = \frac{3}{4}$$

Slope from derivative

$$\frac{dy}{dx} = 3x^2$$

Point(s)

$$3x^2 = \frac{3}{4} \ \rightarrow x = \frac{1}{2} \text{ or } x = -\frac{1}{2}$$

$$\text{When } x = \frac{1}{2} \ \rightarrow y = \frac{8}{9}$$

$$\text{When } x = -\frac{1}{2} \to y = \frac{7}{8}$$

$$\therefore \left(\frac{1}{2}, \frac{8}{9}\right) \text{ and } \left(-\frac{1}{2}, \frac{7}{8}\right)$$

Equations of tangents

$$\text{At } \left(\frac{1}{2}, \frac{8}{9}\right) \text{ the tangent is } y - \frac{8}{9} = \frac{3}{4}\left(x - \frac{1}{2}\right)$$

$$\text{At } \left(-\frac{1}{2}, \frac{7}{8}\right) \text{ the tangent is } y - \frac{7}{8} = \frac{3}{4}\left(x + \frac{1}{2}\right)$$

Category III strategy

First, "put" an arbitrary point on the curve. Use $(y_2 - y_1)/(x_2 - x_1)$ to find the slope of the line through this point and the given point. Now find the derivative. Set the derivative equal to the slope and solve for x to get x-coordinates of your point(s). Put these x-values into the original function to get y-coordinates. You should now have your points. Find the value of the derivative at each of the x-coordinates of each point. You now have a point and a slope for each line.

Category III example

Write an equation(s) of the tangent to $y = x^3 + 1$ that pass through the point $(3, 1)$.

Points

The slope of a line passing through $(x, x^3 + 1)$ and $(3, 1)$ is

$$m = \frac{(x^3 + 1) - 1}{x - 3} = \frac{x^3}{x - 3}$$

$$\text{Now, } \frac{dy}{dx} = 3x^2$$

$$\frac{x^3}{x - 3} = 3x^2 \;\to\; x = \frac{9}{2} \text{ or } x = 0$$

$$\text{When } x = \frac{9}{2} \;\to\; y = \frac{737}{8} \;\to\; \left(\frac{9}{2}, \frac{737}{8}\right)$$

$$\text{When } x = 0 \;\to\; y = 1 \;\to\; (0, 1)$$

Slope of tangents

$$\left.\frac{dy}{dx}\right|_{x=9/2} = \frac{243}{4} \text{ and } \left.\frac{dy}{dx}\right|_{x=0} = 0$$

Equations of tangents

$$y - \frac{737}{8} = \frac{243}{4}\left(x - \frac{9}{2}\right) \text{ and } y - 1 = 0(x - 0)$$

3.5 Differentiability and Continuity

3.5.1 Introduction

Differentiability can be seen as a measure of a curve's smoothness. As you will soon see, a curve that is differentiable is is a smooth curve ... no corners, not cusps, no holes, no vertical asymptotes.

In one sense, the answer to the question, "Is f differentiable at $x = a$?" is easy. All the question asks is "Does the derivative exist at $x = a$?" Consider the following function and its derivative.

$$f(x) = \sqrt[5]{x} \ \rightarrow \ f'(x) = \frac{1}{\sqrt[5]{x^4}}$$

The derivative clearly does not exist at $x = 0$ and so we say that f is not differentiable at $x = 0$. Many times questions about differentiability are this simple ... but not always. There are three situations which are sources for a failure of a function to be differentiable at $x = a$.

- the function may not be continuous at $x = a$
- the function may have a vertical asymptote at $x = a$
- the function may have a corner or cusp at $x = a$.

3.5.2 The continuity problem

If we think of differentiability at a number as the ability to draw a tangent to to the curve at the point, it is clear that if a function is not continuous, it cannot be differentiable. For instance, if $f(a)$ does not exist how are we going to draw a tangent at $x = a$? There's no point there! Similarly, if the left and right limits are not equal at a number, we clearly have to separate pieces that do not meet... a jump discontinuity and we will not be able to draw a tangent there. We do have to be very careful though. Consider the following function.

$$f(x) = \begin{cases} x^2 + 3 & \text{for } x \leq 2 \\ x^2 - 4 & \text{for } x > 2 \end{cases}$$

Is this function differentiable at $x = 2$? The derivative if looks like this.

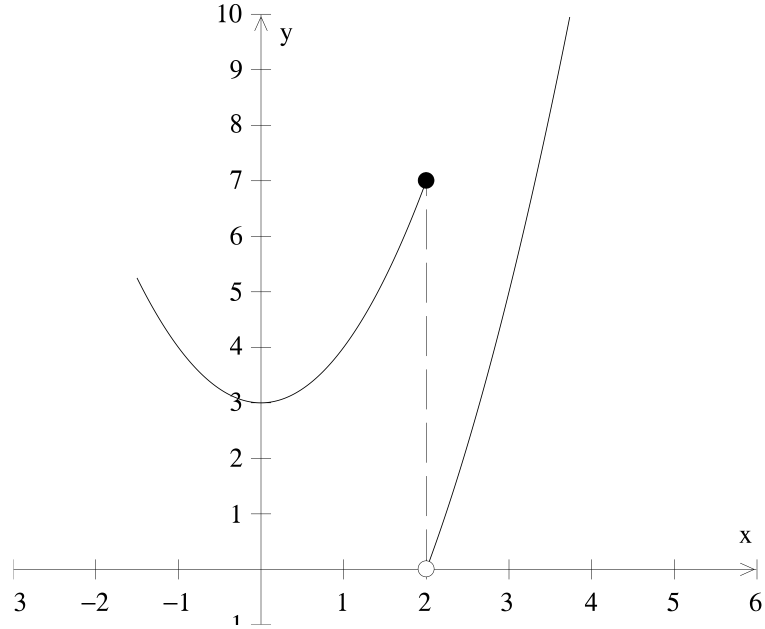
$$f'(x) = \begin{cases} 2x & \text{for } x < 2 \\ 2x & \text{for } x > 2 \end{cases}$$

Notice that we simply took the derivative of each piece. We also dropped the "or equal to" part of the inequality. If we left it in, we're saying that the derivative exists at $x = 2$. But that's what we are trying to determine!

Now to determine if the derivative exists at $x = 2$ we need to find the value of the derivatives from both the left and the right. This should not be surprising–the derivative is a limit and we frequently need to find limits from both the left and right to determine if a limit exists. Here are the values of the derivative and the appropriate notation.

$$f'_+(2) = 4 \text{ and } f'_-(2) = 4$$

Since both of the derivatives have the same value we may be lead to think that the derivative exists at $x = 2$ and thus f is differentiable at $x = 2$. But we'd be mistaken. The problem is that f is not continuous at $x = 2$. The two pieces do not "meet up". The function fails the second part of the continuity test. The limit from the right of 2 is zero but the limit from the left of 2 is 7. The diagram below shows what is happening.



Although the slope of a tangent from the right and left of $x = 2$ are the same, they occur at different points on the curve!

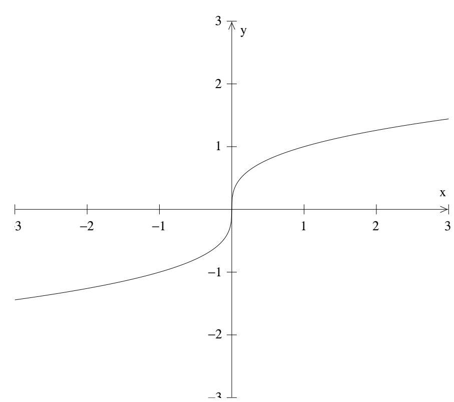
In the end we can say that if a function is discontinuous at $x = a$, then the function is not differentiable at $x = a$.

3.5.3 The vertical asymptote

Consider the following function and its derivative.

$$f(x) = \sqrt[3]{x} \longrightarrow f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

The function exists at $x = 0$ and is continuous at $x = 0$ but its derivative does not exist at $x = 0$. Since the derivative does not exist at $x = 0$, f is not differentiable at $x = 0$. So what's happening at $x = 0$? Take a look at the graph of f.



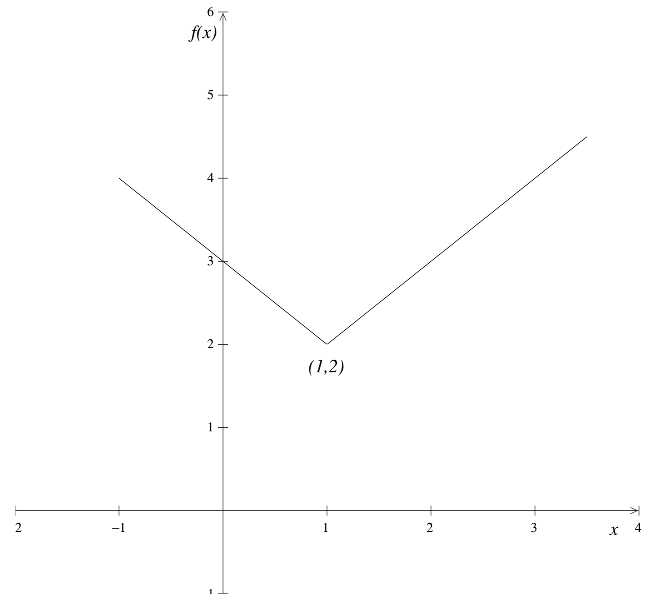
This function has a vertical asymptote at $x = 0$! This will happen almost every time a function exists at $x = a$ but its derivative doesn't.

We know that if a function is not continuous, it is not differentiable. In this case we have a function that is continuous for all x but is not differentiable everywhere. We say that *continuity is a necessary but insufficient condition for differentiability*.

3.5.4 The cusp or corner

The third situation in which a function may fail to be differentiable involves something called a cusp (or corner). Technically these are two are different but we can consider them the same in terms of differentiability.

Consider the function $f(x) = |x - 1| + 2$ and its graph below.



This curve has a corner at $(1, 2)$. This function is continuous at $x = 1$ but is it differentiable? The short answer is no. Note that the slope of a tangent to this curve from the left of $x = 1$ is -1 but the slope of a tangent from the right of $x = 1$ is 1. This means that the derivatives from the left and right are not the same ... thus the function is not differentiable at $x = 1$. The actual mathematics behind this analysis starts with writing the function as a piecewise function and then finding the derivative at $x = 1$.

$$f(x) = \begin{cases} x - 2 + 2 & \text{for } x \geq 1 \\ 1 - x + 2 & \text{for } x < 1 \end{cases}$$

This simplifies to

$$f(x) = \begin{cases} x + 1 & \text{for } x \geq 1 \\ -x + 3 & \text{for } x < 1 \end{cases}.$$

The derivative of f is

$$f'(x) = \begin{cases} 1 & \text{for } x > 1 \\ -1 & \text{for } x < 1 \end{cases}.$$

It's pretty clear that $f'_+(1) = 1$ and $f'_-(1) = -1$ so $f'(1)$ does not exist and thus f is not differentiable at $x = 1$.

Any function that has a cusp or corner will fail to be differentiable at that point.

3.5.5 Summary of relationship between continuity and differentiability

- If a function is differentiable at $x = a$, the function is continuous at $x = a$. *Differentiability implies continuity.*
- If a function is continuous at $x = a$, it may or may not be differentiable at $x = a$.
- If a function is not differentiable at $x = a$, it may or may not be continuous at $x = a$.
- If a function is not continuous at $x = a$, it will not be differentiable at $x = a$. *Not continuous implies not differentiable.*

Now, most of the time, as long as we are not dealing with a piecewise function, the question of differentiability is an easy one. Just take a derivative and see if it exists at the number in question. The only real issue involves piecewise functions.

In dealing with piecewise functions the question becomes, "Should I run a continuity test first?". In the example discussed at the beginning of the "The continuity problem" section, the derivatives from the left and right were equal but the function was not differentiable because it was not continuous at the point in question. We could have avoided testing for differentiability–and being led astray–if we just ran a simple continuity test first.

Here's how to determine if a continuity test is needed. First, on scratch paper, find the derivative. If the value of the derivative from the left and right are different, you can proceed directly to showing the derivative, showing the values of the derivative from the left and right and state that the function is not differentiable. If, on the other had, the value of the derivative from the left and right are the same you should be wary of making any conclusion about differentiability and perform a continuity test first. Sometimes the function turns out to be continuous. If so, show the continuity test, the derivative and its values and conclude the function is differentiable at the number in question. If the function turns out to be discontinuous, you do not need to find the derivative. Just show the continuity test, state that the function is not continuous at the number and therefore the function is not differentiable at that number.

Example 1

Given $f(x) = \begin{cases} x^2 + 1 & \text{for } x \le 1 \\ 2x & \text{for } x > 1 \end{cases}$, determine if f is differentiable at $x = 1$.

We can see fairly quickly that the value of the derivative from the left and the right are equal so we should run a continuity test first.

Continuity test at $x = 1$

$$f(1) = 2$$

$$\lim_{x \to 1^+} f(x) = 2 \text{ and } \lim_{x \to 1^-} f(x) = 2 \therefore \lim_{x \to 1} f(x) = 2$$

Since $f(1) = \lim_{x \to 1} f(x)$, f is continuous at $x = 1$.

Now let's check the derivative.

$$f'(x) = \begin{cases} 2x & \text{for } x > 1 \\ 2 & \text{for } x < 1 \end{cases}$$

$f'_+(1) = 2$ and $f'_-(1) = 2$, $\therefore$ $f'(1)$ exists. Since f is continuous at $x = 1$ and $f'(1)$ exists, f is differentiable at $x = 1$.

Example 2

Given $f(x) = \begin{cases} x^2 + 2 & \text{for } x \le 1 \\ 5x + 3 & \text{for } x > 1 \end{cases}$ determine if f is differentiable at $x = 1$.

We can see fairly quickly that the value of the derivative from the left and the right are not equal so we do not have to run a continuity test!

$$f'(x) = \begin{cases} 2x & \text{for } x < 1 \\ 5 & \text{for } x > 1 \end{cases}$$

$f'_+(1) = 5$ but $f'_-(1) = 2$, $\therefore$ $f'(1) \nexists$ $\therefore$ f is not differentiable at $x = 1$.

Example 3

Given $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ ax + b & \text{for } x \ge 1 \end{cases}$, find the values of a and b so that f is differentiable at $x = 1$.

We are being asked to find the value of two constants, so we need to find two equations in two variables and solve the system. Where will we get the two equations? One from a continuity test and one from differentiability.

Continuity test at $x = 1$

$$f(1) = a + b$$

$$\lim_{x \to 1^+} f(x) = a + b \text{ and } \lim_{x \to 1^-} f(x) = 1$$

For f to be continuous at $x = 1$, $a + b = 1$

Derivative at $x = 1$

$$f'(x) = \begin{cases} 2x & \text{for } x < 1 \\ a & \text{for } x > 1 \end{cases}$$

$$f'_+(1) = a \text{ and } f'_-(1) = 2$$

For f to be differentiable at $x = 1$, $a = 2$

This is a simple system ...just substitute the $a = 2$ into $a + b = 1$ which yields $b = -1$.

Therefore, for f to be differentiable at $x = 1$, $a = 2$ and $b = -1$.

Example 4

Determine if $f(x) = \dfrac{x}{x-5}$ is differentiable at $x = 5$.

Since $f(5)$ does not exist, f is not continuous at $x = 5$ therefore f is not differentiable at $x = 5$.

Example 5

Determine if $g(x) = \sqrt{x+6}$ is differentiable at $x = 10$.

Since $g'(x) = \dfrac{1}{2\sqrt{x+6}} \rightarrow g'(10) = \dfrac{1}{8}$. Since $g'(10)$ exists, g is differentiable at $x = 10$.

3.6 Higher Order Derivatives

3.6.1 Introduction

This particular topic actually contains very little in terms of new concepts although the algebra at times may seem a bit tedious. A "higher order derivative" is simply the derivative of a derivative…or the derivative of a derivative of a derivative…and so on. If you have a function f and find its derivative, you have found f'', the second derivative. For many functions this process can go on forever–and for some other functions the process eventually reaches a point where the derivative is zero or cycles back to the original function. We will see examples of each.

In the course of most of our problems (and calculus in general), the second derivative is as far as we will need to go.

3.6.2 Notation

If we are given a function f, the successive derivatives are denoted:

$$f'(x),\ f''(x),\ f'''(x),\ f^4(x) \ldots f^n(x).$$

If we are given an equation that begins with $y =$, the successive derivatives are denoted:

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4} \cdots \frac{d^ny}{dx^n}$$

If we are dealing with the notation we use for theorems, $D_x[f(x)]$, successive derivatives are denoted:

$$D_x[f(x)],\ D_x^2[f(x)],\ D_x^3[f(x)] \ldots D_x^n[f(x)]$$

3.6.3 Some interesting situations

Consider $f(x) = x^3 + 3x^2 - 2x + 6$. Let's repeatedly differentiate f.

$$\begin{aligned} f'(x) &= 3x^2 + 6x - 2 \\ f''(x) &= 6x + 6 \\ f'''(x) &= 6 \\ f^4(x) &= 0 \end{aligned}$$

Now let's try it again with $f(x) = 2x^4 + x^3 - 5x^2$.

$$\begin{aligned} f'(x) &= 8x^3 + 3x^2 - 10x \\ f''(x) &= 24x^2 + 6x - 10 \\ f'''(x) &= 48x + 6 \\ f^4(x) &= 48 \\ f^5(x) &= 0 \end{aligned}$$

Repeatedly differentiating a polynomial function will eventually lead to a derivative that is zero. In the first example above we began with a third order polynomial and the fourth derivative was zero. In the second, we began with a 4th order polynomial and the fifth derivative was zero. As you have probably guessed, if we begin with an *n*th order polynomial, the $(n+1)$th derivative will be zero.

Let's turn our attention to another interesting scenario. Consider $f(x) = \sin x$ and its successive derivatives.

$$\begin{aligned} f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^4(x) &= \sin x \end{aligned}$$

Clearly the derivatives repeat in a cycle of four. This means that it is very easy to find a very high ordered derivative of $f(x) = \sin x$. Suppose we wanted to know the the 24th derivative. Since the cycle repeats every four derivatives, $f^{24}(x) = \sin x$. If we wanted the 26th derivative, we would just differentiate the 24th derivative two more times. Another way to look at the problem is to ask, "What is the remainder when 26 is divided by 4?" The remainder is two, so the 26th derivative is the same as the 2nd derivative!

The same cyclic behavior occurs with $\cos x$.

$$\begin{aligned} f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^4(x) &= \cos x \end{aligned}$$

Do the other four trigonometric functions exhibit the same cyclic behavior? We leave it to you, the reader, to experiment on your own.

Example 1

Given $y = \frac{1}{x}$, find $\frac{d^3y}{dx^3}$.

$$\begin{aligned} y &= x^{-1} \\ \frac{dy}{dx} &= -x^{-2} \\ \frac{d^2y}{dx^2} &= 2x^{-3} \\ \frac{d^3y}{dx^3} &= -6x^{-4} \\ &= -\frac{6}{x^4} \end{aligned}$$

Example 2

Given $g(x) = (3x+5)^5$, find $g''(x)$.

$$\begin{aligned} g(x) &= (3x+5)^5 \\ g'(x) &= 5(3x+5)^4(3) \\ &= 15(3x+5)^4 \\ g''(x) &= 60(3x+5)^3(3) \\ &= 180(3x+5)^3 \end{aligned}$$

Example 3

Given $f(x) = \sin x$, find $f^{217}(x)$.

217 divided by 4 leaves a remainder of 1 therefore,

$$f^{217}(x) = f'(x)$$

$f'(x) = \cos x$, therefore $f^{217}(x) = \cos x$

Example 4

Given $h(x) = \frac{x-2}{3x+4}$, find $h''(x)$.

$$\begin{aligned} h'(x) &= \frac{(3x+4)(1)-(x-2)(3)}{(3x+4)^2} \\ &= \frac{10}{(3x+4)^2} \end{aligned}$$

Now, at this point we could use the quotient rule again but remember, if the numerator is just a number, bring up the denominator with a negative exponent, then differentiate.

$$
\begin{aligned}
h'(x) &= 10(3x+4)^{-2} \\
h''(x) &= -20(3x+4)^{-3}(3) \\
&= -\frac{60}{(3x+4)^3}
\end{aligned}
$$

Chapter 4

Applications of the Derivative I

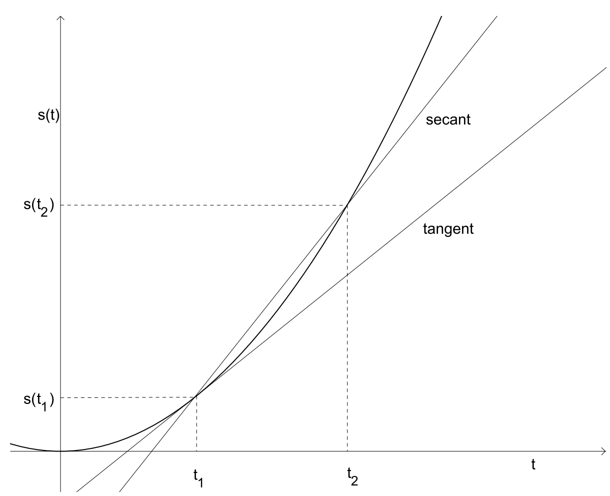
4.1 Rectilinear Motion

4.1.1 Introduction

One of the major reasons calculus was invented was the study of motion. Scientists and engineers were interested in being able to calculate not only the average velocity of an object–the average over a time interval, but also its instantaneous velocity–its velocity at a point in time. In fact, Newton's calculus was very motion-oriented and the analysis of motion was one of his prime interests. This emphasis on motion can be seen in his choice of terms. Newton termed changing quantities *fluents* and their instantaneous rates of change *fluxions*. A fluxion is very close to what we now call a derivative. The study of rectilinear motion (motion in a straight line) is, then, a classic application of calculus.

4.1.2 Average velocity and instantaneous velocity

When we introduced the derivative, we spent some time on graphs of position functions. We learned that the slope of a tangent to the graph of a position function could be interpreted as instantaneous velocity. Consider the graph of a position function $s(t)$ below.



The slope of the secant line, given by

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1},$$

represents the average velocity on (t_1, t_2). It is simply the change in position divided by the change in time on the interval (t_1, t_2). The slope of the tangent at t_1, given by the derivative $s'(t_1)$, represents the instantaneous velocity at t_1, namely $v(t_1)$. Remember, if we take the limit of the difference quotient, we get the derivative.

$$v(t_1) = s'(t_1) = \lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

The derivative of position is velocity.

$$s'(t) = v(t)$$

From now on, when we refer to "velocity" we are usually speaking about instantaneous velocity as opposed to average velocity.

In the course of some problems, our velocity calculations may result in negative quantities. The negative is an indication of direction. By convention, movement to the right or up is positive and movement down or to the left is negative. Thus, if we are working on a problem about an object being thrown vertically upward into the air, a positive velocity at a particular time indicates the object is moving upwards. A negative velocity would indicate the object is moving downward. *Speed* is the absolute value of velocity.

A note about giving answers: If you calculate a velocity and the result is negative, like -64 feet per second, it is preferable to answer the question by saying, "Therefore the velocity is 64 feet per second downward." rather than, "Therefore the velocity is -64 feet per second." It is important that you interpret your result in you answer.

4.1.3 Acceleration

We have determined that a change in position divided by a change in time is velocity. What happens when we divide a change in velocity by a change in time? If we in a car and our velocity is changing over time, we are accelerating. The *average acceleration* of an object can be given by

$$\frac{v(t_2) - v(t_1)}{t_2 - t_1}.$$

If we then take a limit as $t_2 \to t_1$, we would have the instantaneous acceleration at t_1. Again, when we use the term acceleration in this course, we will usually mean instantaneous acceleration at a point in time.

The derivative of velocity is acceleration.

$$s'(t) = v(t) \;\rightarrow\; v'(t) = a(t)$$

We can now address the issue of "speed". Speed is the absolute value of velocity…that much is easy. To determine if the speed of an object is *increasing or decreasing* is another matter and depends on both velocity *and* acceleration. It will help to think of acceleration as a force acting on the object. There are four scenarios which can occur.

- If $v > 0$ and $a > 0$, the object is moving to the right and being pushed to the right so its velocity is increasing and its speed is increasing.
- If $v > 0$ and $a < 0$, the object is moving to the right and being pushed to the left so its velocity is decreasing and its speed is decreasing.
- If $v < 0$ and $a > 0$, the object is moving to the left and being pushed to the right so its velocity is increasing (getting less negative) and its speed is decreasing (it is slowing down in the negative direction).
- If $v < 0$ and $a < 0$, the object is moving to the left and being pushed to the left so its velocity is decreasing (getting more and more negative) and its speed is increasing (it is speeding up in the negative direction).

Notice that anytime the velocity and acceleration have the same sign, the speed is increasing and when velocity and acceleration have opposite signs, the speed is decreasing.

A word on notation: We will see position functions written several ways. Many times, if an object is moving horizontally, the position function will be written as "$x(t) =$". Position functions can also be written as "$s(t) =$", "$y =$" or "$y(t) =$".

Example 1

A particle is moving so that its position is given by $s(t) = 2t^3 - 4t^2 + 2t - 1$. Determine the intervals of time when the particle is moving to the right and the intervals on which the particle is moving to the left. Determine when the particle is at rest and when the particle changes direction.

We begin with the derivative.

$$v(t) = 6t^2 - 8t + 2.$$

$$v(t) \exists \text{ for all } t \text{ and } v(t) = 0 \text{ when } t = \frac{1}{3} \text{ or } t = 1.$$

Now, because the graph of the velocity function is parabola, it is easy to determine where it is positive and negative. It is also clear from a quick sketch at what values the particle changes direction.

$$\text{Since } v(t) > 0 \text{ on } \left(-\infty, \frac{1}{3}\right) \cup (1, \infty)\text{, the particle is moving to the right on } \left(-\infty, \frac{1}{3}\right) \cup (1, \infty).$$

$$\text{Since } v(t) < 0 \text{ on } \left(\frac{1}{3}, 1\right)\text{, the particle is moving to the left on } \left(\frac{1}{3}, 1\right).$$

$$\text{Since } v(t) = 0 \text{ at } t = \frac{1}{3} \text{ and } t = 1\text{, the particle is at rest at } t = \frac{1}{3} \text{ and } t = 1.$$

Since $v(t) > 0$ on $\left(-\infty, \frac{1}{3}\right)$ and $v(t) < 0$ on $\left(\frac{1}{3}, 1\right)$, the particle changes direction at $t = \frac{1}{3}$.

Since $v(t) < 0$ on $\left(\frac{1}{3}, 1\right)$ and $v(t) > 0$ on $(1, \infty)$, the particle changes direction at $t = 1$.

Example 2

A ball is thrown vertically upward from the ground with an initial velocity of 64 feet per second. If positive is up, the ball's position is given by $s(t) = -16t^2 + 64t$, where t is measured in seconds and position in feet. (a) Is the ball rising or falling at $t = 1$? (b) Is the ball rising or falling at $t = 3$? (c) With what velocity does the ball strike the ground? (d) How high does the ball go?

We begin by finding the velocity function: $v(t) = -32t + 64$.

(a) Since $v(1) = 32$ the velocity of the ball at $t = 1$ is 32 feet per second. Since $v(1) > 0$, the ball is rising.

(b) Since $v(3) = -32$ the velocity of the ball at $t = 1$ is -32 feet per second. Since $v(1) < 0$, the ball is falling.

(c) First we need to determine when the ball hits the ground. We do this by setting the position equal to zero. $s(t) = 0 \;\rightarrow\; t = 0$ or $t = 4$. The zero represents the position of the ball at the start of the problem, so we will use $t = 4$. Since $v(4) = -64$ the velocity of the ball at impact is 64 feet per second downward. (Note how we interpreted the negative value in our answer.)

(d) At the maximum height, $v(t) = 0 \;\rightarrow\; t = 2$. Since $s(2) = 64$ the maximum height the ball reaches is 64 feet.

4.2 Implicit Differentiation

4.2.1 Introduction

Equations can be either *explicitly* or *implicitly* defined. An equation like $f(x) = x^3$ or $y = \sin x$ or $s = t^2 - t$ are explicitly defined. The dependant variable is explicitly defined in terms of the dependent variable. Equations like $x^2 + y^3 = 4$ or $x^2y^2 = 7xy$ are implicitly defined. Exactly what the independent variable is in terms of the independent variable is not clearly defined ...it is implied. In an implicitly defined equation, we imply that one variable is a function of the other but never state it outright. We can even have an equation in which all the variables are a function of a variable not present in the equation. For instance, in the equation $x^2 + y^2 = 9$, both the x and the y could be changing over time. Thus, the x and y are functions of time t.

Differentiating implicitly defined equations is actually just an application of the chain rule. If we were to differentiate the expression $[f(x)]^2$, we use the chain rule and get $2f(x)f'(x)$. Now, if we just replace the $f(x)$ with a y the derivative would look like

$$2y\,\frac{dy}{dx}$$

where $\frac{dy}{dx}$ is the derivative of y with respect to x. If you always remember that implicit differentiation is simply an application of the chain rule, you should have no problem.

Example 1

Determine $D_x\left[y^7\right]$

By asking us to differentiate with respect to x, we are being told that the y is a function of x. If we think of y as a function of x, the problem becomes $D_x\left[(f(x))^7\right]$. Applying the chain rule would yield

$$D_x\left[(f(x))^7\right] = 7\left[f(x)\right]^6 f'(x)$$

Now, we can differentiate the expression directly and get

$$D_x\left[y^7\right] = 7y^6\frac{dy}{dx}$$

Example 2

Determine $D_m\left[b^7\right]$.

The question implies that b is a function of m, therefore,

$$D_m\left[b^7\right] = 7b^6\frac{db}{dm}$$

Note: $\dfrac{db}{dm}$ is the derivative of b with respect to m.

Example 3

Given $x^2 + y^5 = 9$, find $\dfrac{dy}{dx}$.

The implication here is that y is a function of x so terms that involve x will be differentiated "normally" and y terms will be chained.

$$x^2 + y^5 = 9$$

$$2x + 5y^4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{5y^4}$$

Example 4

Given $y^5 + 3x^2y^2 + 5x^4 = 12$, find $\dfrac{dy}{dx}$.

Note that to differentiate the middle term on the left, we will need to use the product rule.

$$\begin{aligned}
y^5 + 3x^2y^2 + 5x^4 &= 12 \\
5y^4\frac{dy}{dx} + \left[(3x^2)\left(2y\frac{dy}{dx}\right) + (y^2)(6x)\right] + 20x^3 &= 0 \\
5y^4\frac{dy}{dx} + 6x^2y\frac{dy}{dx} + 6xy^2 + 20x^3 &= 0 \\
\frac{dy}{dx}\left[5y^4 + 6x^2y\right] &= -20x^3 - 6xy^2 \\
\frac{dy}{dx} &= -\frac{20x^3 + 6xy^2}{5y^4 + 6x^2y}
\end{aligned}$$

Example 5

Given $u^4 + v^3 = 8uv$, find $\frac{dv}{du}$.

In this problem, since we are asked to find $\frac{dv}{du}$, the v is a function and the u is our variable.

$$\begin{aligned}
u^4 + v^3 &= 8uv \\
4u^3 + 3v^2\frac{dv}{dv} &= 8u\frac{dv}{du} + 8v \\
\frac{dv}{du} &= \frac{8v - 4u^3}{3v^2 - 8u}
\end{aligned}$$

Example 6

Given $x = \sin y$, find $\frac{dy}{dx}$.

$$\begin{aligned}
1 &= \cos y\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{1}{\cos y} \\
\frac{dy}{dx} &= \sec y
\end{aligned}$$

Now that we have implicit differentiation, we never have to solve an equation like $x^2 + y = xy$ for y before differentiating. We just implicitly differentiate. As a matter of fact, you should not attempt to solve for a particular variable before differentiation–many times it is impossible and other times it can lead you astray.

We can differentiate any equation with respect to any variable–even one that is not in the problem. We will do this frequently in our next section on related rates.

Example 7

Differentiate $x^3 + y^2 = 9$ with respect to t.

In this case, both the x and the y are functions (and must be chained) of t.

$$3x^2\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

There is nothing left to do here . . . we have differentiated with respect to t so we are finished.

4.3 Related Rates

4.3.1 Introduction

Related rate problems are applications of implicit differentiation. The are one of the core of basic applications presented in a first year calculus course. No one can take calculus without encountering related rate problems. “Word problems” begin in elementary school with, “Mary had four apples. . . ” and end in calculus with related rate problems. Have fun! These are genuinely *tasty* problems!

The problems ask us to determine quantities and/or rates of change in quantities in situations involving several variables. They are called *related* rate problems because one or more rates are changing in the problem and as one changes, one or more others change. . . thus *related* rates. Because they are rate problems, all the variables in all the problems will be functions of t. . . therefore all rates are $\dfrac{d\ something}{dt}$.

The good news is that we already know all the mathematics to solve these problems. Like all word problems, the main task is in setting up a solution. Here are some general guidelines to follow:

- Draw a picture when possible. Label all elements of the diagram. If a particular distance is changing, label it with a variable–even if a numerical value is given for some point in time. Avoid putting numbers on a diagram unless the distance is a constant–like the length of a ladder or the width of a river.
- Write down all the given information. If we’ve labeled a particular segment x and the problem states that it is increasing at 5 feet per second, we are being told that $\dfrac{dx}{dt} = 5$.
- Write down the problem statement. Generally it will sound something like, “Find $\dfrac{dx}{dt}$ when $y = 4$ and $x = -3$.”
- When writing down the rates given in the problem, if the length is getting smaller, the rate needs to be negative. If the length is getting larger the rate is positive.
- Write an equation that ties together the variables in the problem. Many times this will be the Pythagorean theorem or a known area or volume formula. Do not put in any numbers yet!
- Implicitly differentiate your equation with respect to t.
- After differentiating, substitute any information given in the problem and solve for the required variable or rate.

- Units belong in your answer only! Do not put units on your mathematics.

4.3.2 Categories of related rate problems

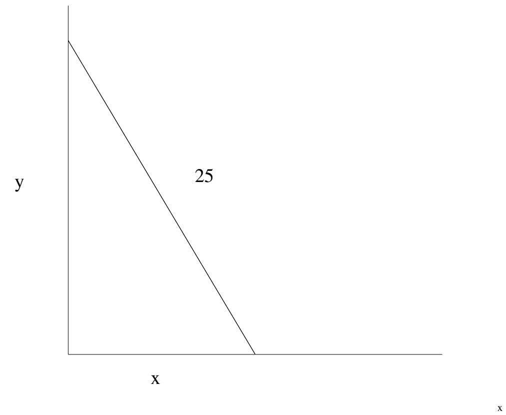
Although there are on infinite variety of related rate problems, there are several types which can be found in every calculus text published since L'Hopital put his text on the bookshelf. A short list is given below with a suggestion for setting up an equation so solve the problem.

- The Ladder Problem: Generally, a ladder leans against a wall and typically the bottom is pulled out at a constant rate. This almost always makes use of the Pythagorean theorem. (Likely to be voted most famous related rate problem!)
- The Shadow Problem: Involves a person walking toward or away from a suspended light. Using similar triangles almost always works for this one.
- The Kite Problem: Susie flies a kite at a constant height as she lets the string out. Use the Pythagorean theorem.
- The Intersection Problem: Lots of variety here but basically involves two cars moving toward or away from an intersection. Use the Pythagorean theorem.
- The Sphere Problem: Snowballs, ball bearings, melting ice on a dome–use the formula for the volume of a sphere.
- The Searchlight Problem: A searchlight pointed at a plane which flies directly over the light. Most often this problem involves use trigonometric functions.
- The Cone Problem: Voted most difficult by students world-wide. Usually involves some sort of liquid being poured into an inverted right circular cone. Use the formula for the volume of a cone. You will almost always have to use similar triangles to eliminate the r in the formula before differentiating.

The only way to learn and become proficient at related rate problems is to do lots of them.

Example 1

A ladder 25 feet long leans against a vertical wall. If the bottom of the ladder is pulled away from the wall at a constant rate of 3 feet per second, how fast is the top of the ladder sliding down the wall when the bottom is 15 feet from the base of the wall?



$\dfrac{dx}{dt} = 5$ because the x is changing at 5 feet per second. (If the bottom was being pushed toward the wall, we would have called it -5 because the x would be getting smaller.

We need to find $\dfrac{dy}{dy}$ when $x = 15$

$$x^2 + y^2 = 25$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Now substitute all the known values ... including the length of y when the x is 15 which you can get by using the Pythagorean theorem.

$$2(15)(3) + 2(20)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{9}{5}$$

Therefore the top of the ladder is sliding down the wall at $\dfrac{9}{5}$ feet per second.

Note that we interpreted the negative rate in our answer.

Example 2

A spherical balloon is being inflated so that its volume is increasing at the rate of 5 cubic meters per minute. At what rate is the diameter increasing when the diameter is 12 meters.

The volume of a sphere is given by $V = \dfrac{4}{3}\pi r^3$.

Since we are doing this problem in terms of the diameter we will substitute $d/2$ for r.

$$V = \frac{\pi}{6}d^3$$

We need to find $\dfrac{dd}{dt}$ when $d = 12$.

$$\frac{dV}{dt} = \frac{\pi}{2}d^2\frac{dd}{dt}$$

$$5 = \frac{\pi}{2}12^2\frac{dd}{dt}$$

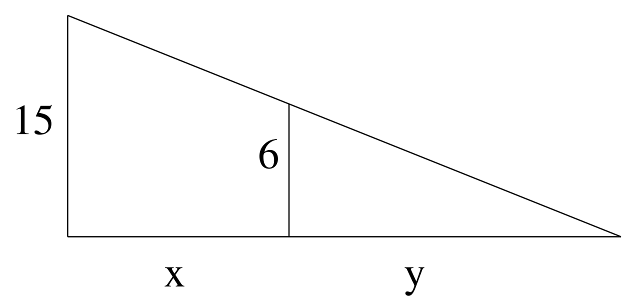
$$\frac{dd}{dt} = \frac{5}{72\pi}$$

Therefore the diameter is increasing at $\frac{5}{72\pi}$ meters per minute.

Example 3

A light is hung 15 feet above a straight path. If a man 6 feet tall is walking away from the light at a rate of 5 feet per second, how fast is his shadow lengthening when he is 20 feet from the light? How fast is the tip of his shadow moving when he is 20 feet from the light?

Most shadow problems can be done using similar triangles.



In our diagram, x is the horizontal distance from the man to the point directly below the light and y is the length of his shadow.

We are given that $\frac{dx}{dt} = 5$ and we need to find $\frac{dy}{dt}$ when $x = 20$.

The large triangle with 15 as its vertical side and the smaller triangle with 6 as its vertical side are similar so the following relationship can be set up:

$$\frac{15}{x+y} = \frac{6}{y}$$

Cross multiplying and simplifying yields: $15y = 6x + 6y$

$$3y = 2x$$

Now, differentiate with respect to t.

$$3\frac{dy}{dt} = 2\frac{dx}{dt}$$

For $\dfrac{dx}{dt} = 5$ we obtain $\dfrac{dy}{dt} = \dfrac{15}{2}$.

Therefore, his shadow in getting larger at $\dfrac{15}{2}$ feet per second.

Now, the tip of his shadow is moving for two reasons. . . he is moving and the shadow itself is growing. This means that the tip is moving at the speed he is walking plus the speed at which the shadow is growing. Thus the tip of the shadow is moving at

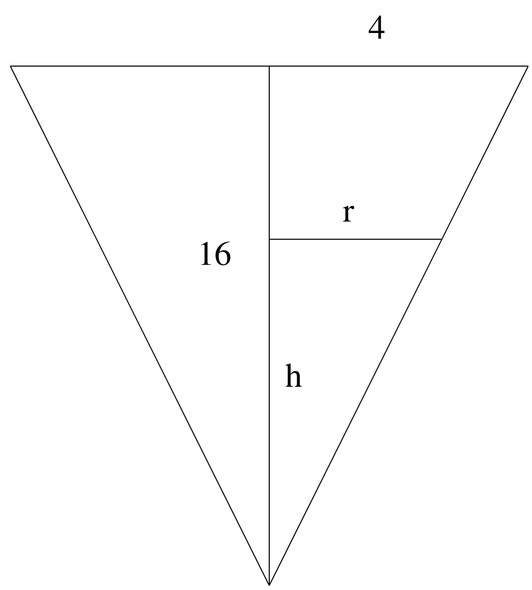
$$\frac{dx}{dt} + \frac{dy}{dt} = \frac{25}{2}.$$

Therefore the tip is moving at $\dfrac{25}{2}$ feet per second.

Note: In this problem (as all shadow problems), his original distance from the light has no impact on how fast the shadow is growing or the speed with which the tip is moving.

Example 4

A tank is in the form of an inverted cone having an altitude of 16 meters and a radius of 4 meters. Water is flowing into the tank at a rate of 2 cubic meters per minute. How fast is the water level rising when the water in the tank is 5 meters deep?



One of the keys to inverted cone problems is to get the radius of the cone expressed in terms of the height. This can be done with similar triangles.

$$\frac{16}{h} = \frac{4}{r} \;\rightarrow\; r = \frac{h}{4}$$

The volume of an inverted cone is

$$V = \frac{1}{3}\pi r^2 h.$$

For $r = \dfrac{h}{4}$ the formula becomes

$$V = \frac{\pi}{48}h^3.$$

Differentiating with respect to t yields $\dfrac{dV}{dt} = \dfrac{\pi}{16}h^2\dfrac{dh}{dt}$.

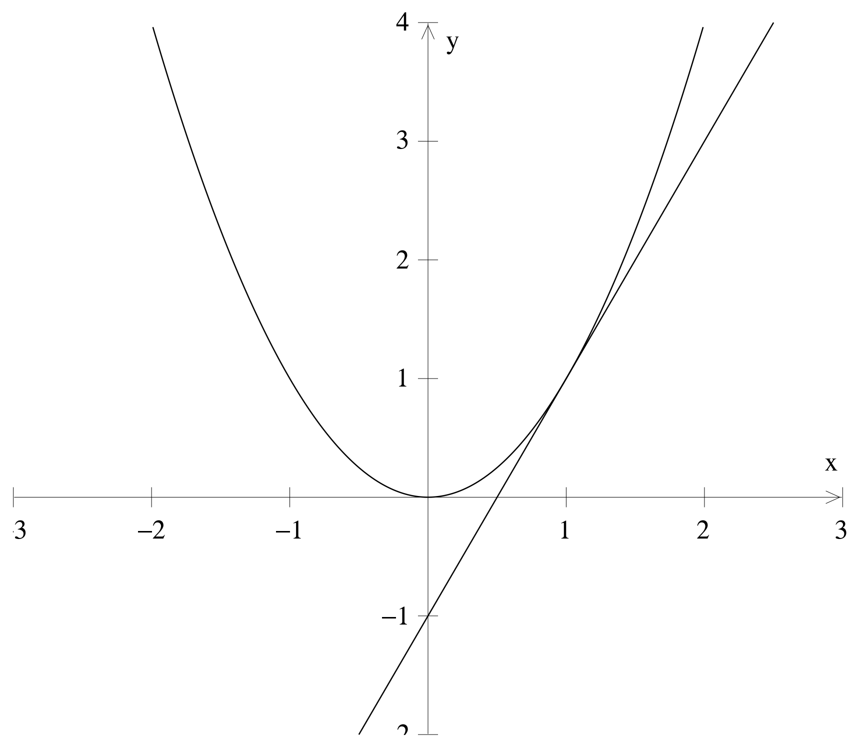
For $h = 5$ and $\dfrac{dV}{dt} = 2 \;\rightarrow\; \dfrac{dh}{dt} = \dfrac{32}{25\pi}$

Therefore the height of the water is increasing at $\dfrac{32}{25\pi}$ meters per minute.

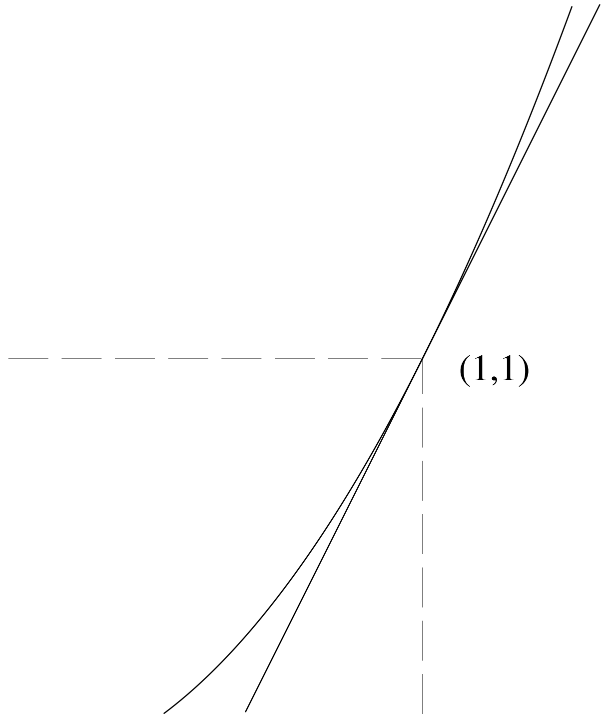
4.4 Local Linearity and Linearizations

4.4.1 Local linearity

Take a look at the graph of $f(x) = x^2$ and the tangent line to f at $x = 1$.



Now, let's look even closer at the region of the graph very near to the point of tangency, $(1, 1)$.



Notice how little difference there is between the function $f(x) = x^2$ and the tangent line $y = 2x - 1$ very close to $x = 1$. Because there is so little difference between the function and the tangent line near $x = 1$, we could get very good estimates of function values near $x = 1$ by finding values of $2x - 1$ instead of x^2! Take a look at the table below:

x	$f(x)$	$2x - 1$
1.1	1.210	1.200
1.01	1.020	1.020
1.001	1.002	1.002
.9	.810	.800
.95	.903	.900
.99	.980	.980

Note that as we get closer and closer to $x = 1$ (from the left and the right) values for $f(x) = x^2$ and $y = 2x - 1$ get very close. In other words, for values of x close to 1, the tangent line can be used to approximate function values of $f(x) = x^2$! This process of replacing a non-linear function with a linear function is called a linearization of f at $x = a$.

This illustrates the basic idea of local linearity. . . that all differentiable functions are locally linear. Around specific values of x, functions can be replaced by tangent lines! It's easier to "plug" a number into a linear function than a non-linear function. . . that's the value in the whole idea.

Now, you may have already noticed that all we did to linearize $f(x) = x^2$ at $x = 1$ was to write an equation of a tangent to f at $x = 1$. You're right. . . that's all we did and that's all a linearization is.

We do however, like to write linearizations as functions instead of equations.

Consider the point-slope form of a line.

$$y - y_1 = m(x - x_1)$$

Solving for y yields

$$y = y_1 + m(x - x_1).$$

Now, if we need the tangent at $x = a$, the slope (m) is $f'(a)$ and $y_1 = f(a)$. Making these substitutions yields

$$y = f(a) + f'(a)(x - a).$$

Now, we just call this function $L(x)$ instead of y.

$$L(x) = f(a) + f'(a)(x - a).$$

In general, the linearization of f at $x = a$ is written

$$L(x) = f(a) + f'(a)(x - a)$$

Example 1

Linearize $f(x) = x^3 - x$ at $x = 2$, then use the linearization to approximate $f(2.1)$.

Since $f(2) = 6$ and $f'(x) = 3x^2 - 1 \longrightarrow f'(2) = 11$ we get

$$L(x) = 6 + 11(x - 2)$$

Now, $f(2.1) \approx L(2.1) = 7.1$

(The actual value of $f(2.1)$ is 7.161.)

Example 2

Use a linearization to approximate the value of $\sqrt{49.1}$.

Let $f(x) = \sqrt{x}$ and linearize f at $x = 49$.

$$f(49) = 7 \text{ and } f'(x) = \frac{1}{2\sqrt{x}} \longrightarrow f'(49) = \frac{1}{14}$$

$$L(x) = 7 + \frac{1}{14}(x - 49)$$

$$\sqrt{49.1} \approx L(49.1) = 7 + \frac{1}{14}(49.1 - 49) = 7.007$$

Note: A linearization is actually something called a "first order Taylor polynomial". Taylor polynomials are used to replace non-polynomial functions to whatever degree of accuracy is needed.

4.4.2 Error in linearizations

There are times when we will want to know the accuracy of our linearization. This question is usually stated as "for what interval of x values is the given linearization accurate to within .5 units". This question asks us what values we can put into our linearization and still get function value estimates that are within .5 of the actual value.

If the allowable error is denoted as E, we simply need to solve the inequality

$$f(x) - E < L(x) < f(x) + E.$$

Graphically we are looking for the interval where the graph of our linearization lies between the graph of the function shifted E units up and E units down.

Example 3

For what values will the linearization of $f(x) = \sqrt{x}$ at $x = 25$ be accurate to within .1 units?

The linearization of $f(x) = \sqrt{x}$ at $x = 25$ is $L(x) = 5 + \frac{1}{10}(x - 25)$.

We need to solve the inequality:

$$\sqrt{x} - .1 < 5 + \frac{1}{10}(x - 25) < \sqrt{x} + .1$$

Now, these type of inequalities are normally solved with a computer or calculator. In this case the solution is

$$16 < x < 36.$$

This means that we can get square roots of any number between 16 and 36 by plugging the number into $5 + \frac{1}{10}(x - 25)$ instead of actually taking a square root and still get values that are within .1 of the actual square root!

For instance, the square root of 30 can be obtained by $5 + \frac{1}{10}(30 - 25) = 5 + .5 = 5.5$

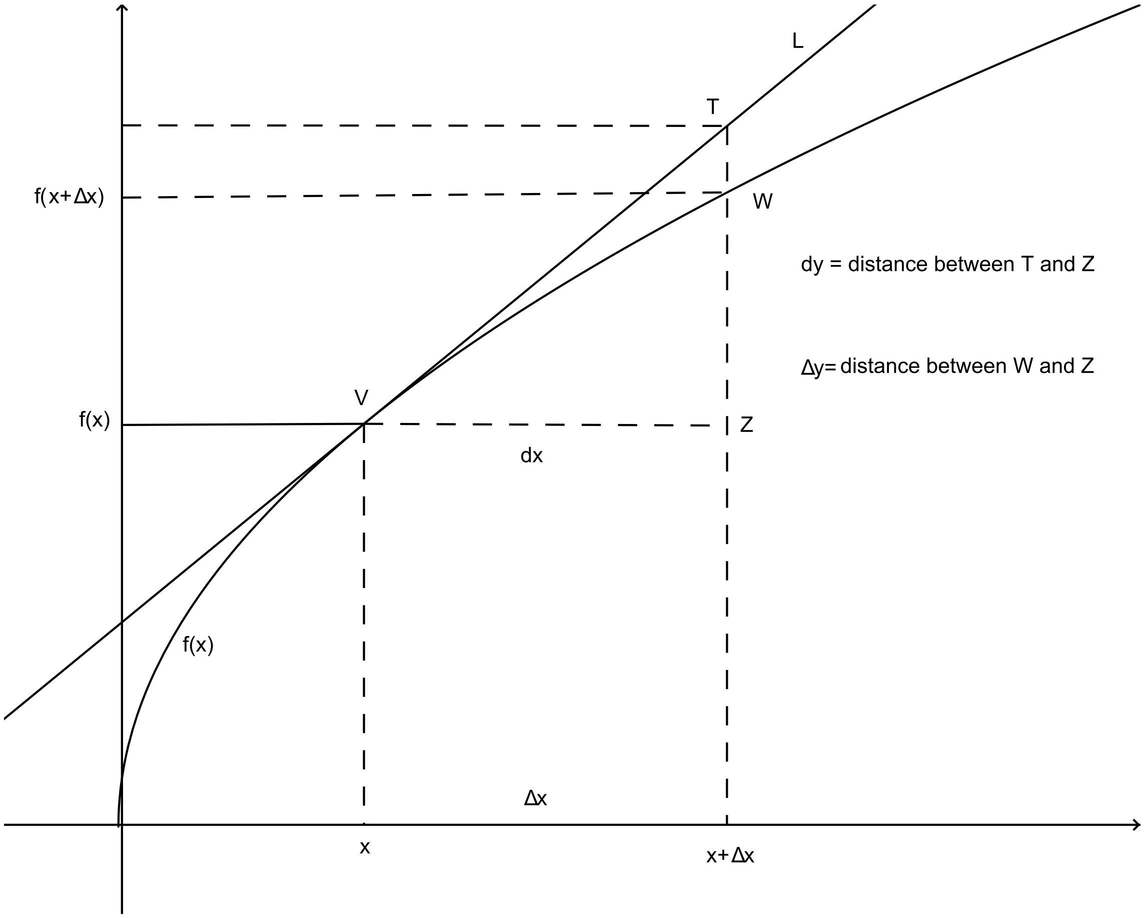
The actual value (to three decimal places) of $\sqrt{30}$ is 5.477. Pretty good huh?!

4.5 Differentials

4.5.1 Introduction

Differentials are closely related to linearizations–they are a natural consequence of local linearity. Many problems that can be done with linearizations can also be done with differentials.

Up until now we have used $\frac{dy}{dx}$ as a symbol for the derivative of y with respect to x. The reality is that $\frac{dy}{dx}$ is actually a fraction. The dx and the dy are called differentials and are real numbers. To see this, take a look at the diagram below.



The slope of the tangent line L is given by the derivative,

$$\frac{dy}{dx}.$$

The slope of any line is given by

$$\frac{rise}{run}.$$

Thus, the dx is the "run" and the dy is the "rise". In the diagram above, dx is the distance from point V to point Z. It is the same as Δx. Similarly, dy is the distance from point Z to point T.

As our input goes from x to $x + \Delta x$, the output changes from $f(x)$ to $f(x + \Delta x)$. This *exact change in function values* is called Δy and is given by

$$\Delta y = f(x + \Delta x) - f(x).$$

Now, take a look at the distance between point T and point W. This distance is the difference between dy and Δy. Note that as dx (or Δx) decreases, the difference between dy and Δy grows smaller and smaller. This is the usefulness of differentials! For very small changes in inputs, dy is almost the same as Δy. In other words

$$dy \approx \Delta y, \text{ for small } \Delta x's$$

(Remember that $dx = \Delta x$.)

Since dx and dy are actually separate entities, we can separate them! We know that $\frac{dy}{dx}$ and $f'(x)$ are both ways to write a derivative. This means that

$$\frac{dy}{dx} = f'(x) \longrightarrow dy = f'(x)dx$$

As an example, if

$$y = x^2 + \sin x \longrightarrow \frac{dy}{dx} = 2x + \cos x \longrightarrow dy = (2x + \cos x)\, dx$$

So what do differentials do for us? They allow us to estimate changes in functions values. As you remember, linearizations allow us to approximate *function values* . . . differentials allow us to approximate *changes in function values*.

Let's see how they work.

Example 1

Given $y = x^3 + 2x^2$. For $x = 2$ and $dx = .1$, find Δy and then find dy.

$$\Delta y = f(x + \Delta x) - f(x)$$

For $x = 2$ and $\Delta x = .1$,

$$\Delta y = f(2.1) - f(2) = 2.081$$

$$\frac{dy}{dx} = 3x^2 + 4x \longrightarrow dy = (3x^2 + 4x)dx$$

For $x = 2$ and $dx = .1 \longrightarrow dy = (20)(.1) = 2$

This means that our approximation for the change in y is only 81 thousandths different from the actual change in y.

The value in using differentials (like linearizations) is that it is easier and quicker to use differentials.

Example 2

Use differentials to find the volume of glass needed to make a hollow sphere whose inner radius is 2 inches and is to be .01 inches thick.

We could just do this problem by subtracting the volume of the hollow core (radius 2 inches) from the total volume of the sphere (radius 2.1 inches). This would require us to calculate two volumes. Differentials allow us a much quicker calculation.

$$V = \frac{4}{3}\pi r^3 \longrightarrow dV = 4\pi r^2\, dr$$

For $r = 2$ and $dr = .1$ we get $dV = .503$

Thus the approximate change in volume (the volume of the glass itself) is approximately .503 cubic inches.

As was previously stated, there are many problems that can be done with either linearizations or differentials.

Example 3

Use an appropriate linearization to estimate the value of $\sqrt{81.7}$. Then use differentials to perform the same estimate.

Linearization

We will linearize $f(x) = \sqrt{x}$ at $x = 81$.

$$f(81) = 9 \text{ and } f'(x) = \frac{1}{2\sqrt{x}} \longrightarrow f'(81) = \frac{1}{18}$$

$$L(x) = 9 + \frac{1}{18}(x - 81)$$

$$L(81.7) = 9 + \frac{1}{18}(81.7 - 81) \longrightarrow L(81.7) = 9 + \frac{7}{180} \approx 9.039$$

$$\therefore \sqrt{81.7} \approx 9.039$$

Differentials

$$\text{Since } y = \sqrt{x} \longrightarrow dy = \frac{1}{2\sqrt{x}}\,dx$$

$$\text{For } x = 81 \text{ and } dx = .7 \longrightarrow dy = \frac{7}{180}$$

$$\therefore \sqrt{81.7} \approx 9 + \frac{7}{180} \approx 9.039$$

Note that differentials only give us the approximate change in function values. This change must be added to an original function value. In the case the original value is $\sqrt{81} = 9$.

Note also that when we used linearizations, the $7/180$ was also added to the 9 to get our estimate.

4.6 L'Hopital's Rule

4.6.1 Introduction

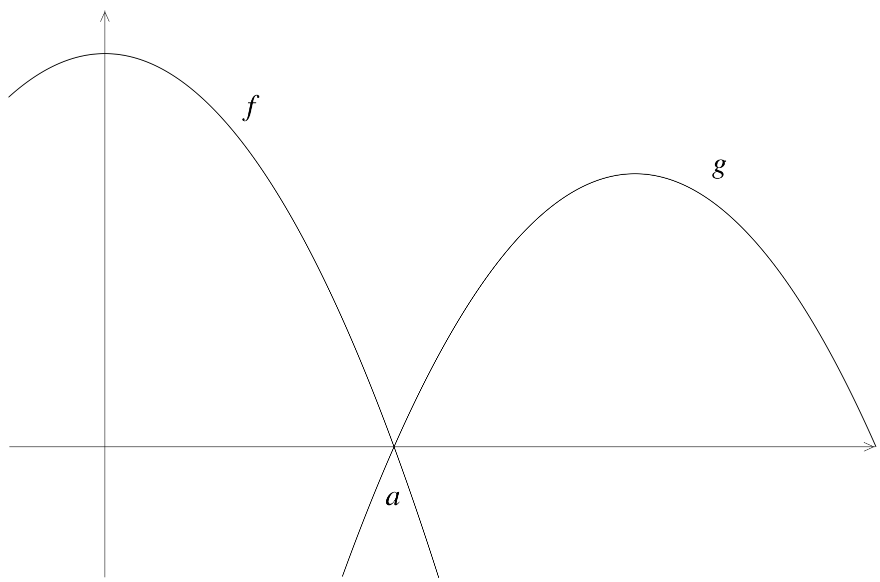
L'Hopital's rule is a theorem used to find limits when the result of "plugging in the a" is of indeterminant form. There are several indeterminant forms in mathematics but the one that concerns us is 0/0. We have chosen to introduce L'Hopital's rule now because local linearity can be used to derive it. Once we have L'Hopital's rule, we will no longer ever have to factor or rationalize to find limits.

A little history . . .
Although the theorem is called "L'Hopital's" rule, the Marquis de L'Hopital (1661-1704)did not actually come up with it. It was discovered by the Swiss mathematician Johann Bernoulli (1667-1748), one of

the famous Bernoulli family. Johann was actually the 10th child of his parents. Johann met L'Hopital in Paris and L'Hopital asked Bernoulli to teach him the new differential calculus that just been published by Liebnitz. L'Hopital went on to publish the very first textbook on differential calculus–in which he included the rule now known as "L'Hopital's rule". Although L'Hopital did thank Bernoulli in the preface to the book, he made no mention of what material he actually learned from Bernoulli. In fact, L'Hopital paid Bernoulli a fair sum of money for his input and part of the condition of the payment was that Bernoulli would say nothing until after L'Hopital's death. In 1922, it was finally shown that Johann Bernoulli did in fact discover the rule . . . and much of the other work found in L'Hopital's text.

4.6.2 Derivation of L'Hopital's rule

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of indeterminant form $\left(\frac{0}{0}\right)$, it means that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Graphically, the situation is shown below.



The idea now is to linearize both f and g at $x = a$. The linearization of f becomes $L_f(x) = f(a) + f'(a)(x - a)$ and the linearization of g at $x = a$ becomes $L_g(x) = g(a) + g'(a)(x - a)$.

We can now replace the actual functions in the limit with their linearizations. We can do this because the linearizations can replace the functions as long we we stay very close to a . . . which we are doing by taking a limit as $x \to a$. We now have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}.$$

We know that $f(a) = 0$ and $g(a) = 0$ so

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)}.$$

We can now reduce the common factor $(x - a)$.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)}{g'(a)}.$$

Because we are letting $x \to a$, this expression can be written:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

L'Hopital's Rule

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of indeterminant form them

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

You can use L'Hopital's rule repeatedly. If you get a 0/0, use L'Hopital, plug your number in again and get 0/0 again, use L'Hopital again. You can use it as long as you are getting the 0/0.

Example 1

Find $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$.

Since this limit results in a 0/0 we can apply L'Hopital's rule.

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{6}$$

Example 2

Find $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$.

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{2x}{1} = 10$$

Example 3

Find $\lim_{x \to 0} \frac{\sin x}{x}$.

We already know this limit but look how nice it works with L'Hopital!

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

Example 4

Find $\lim_{x\to 0}\frac{\sin x}{x^2}$.

$$\lim_{x\to 0}\frac{\sin x}{x^2} = \lim_{x\to 0}\frac{\cos x}{2x} \nexists$$

This time, the original limit yielded 0/0, we applied L'Hopital, then got a 1/0 . . . thus the limit fails to exist.

Just remember, you can only use L'Hopital's rule if the result of "plugging in the a" is 0/0.

4.7 Newton's Method

4.7.1 Introduction

Newton's Method is a method used to approximate the zeros of a function. Again, like differentials and L'Hopital's rule, it is a consequence of and relies on the concept of local linearity.

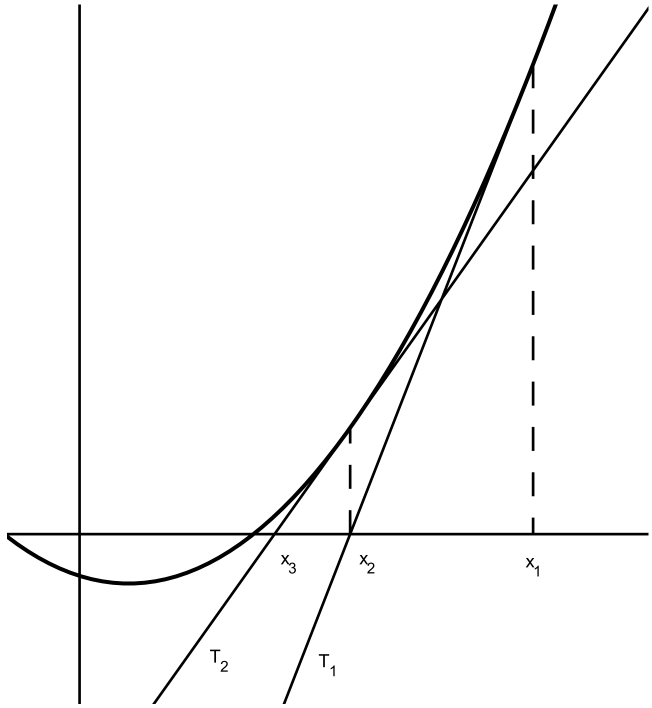
4.7.2 The process

Newton's method is an example of an iterative process. An iterative process uses the output from a calculation as an input. For example, if we used x^2 iteratively with an initial input of 2 we would get 4 as a first output, then put the 4 back into x^2 and get a 16, then put the 16 into x^2, etc.

Newton's method proceeds like this:

- An initial guess (x_1) is made for the zero.
- The function is linearized at at x_1.
- The x-intercept of this linearization is calculated and called x_2.
- The function is linearized at x_2–this is the second approximation of the zero.
- The function is again linearized–this time at x_2.
- This process is repeated until the desired level of accuracy is obtained.

The process can be seen in the diagram below.



4.7.3 Derivation of formula for Newton's method

If we linearized a function f at $x = x_1$ the slope would be $f'(x_1)$ and the point of tangency would be $(x_1, f(x_1))$. The equation of the first tangent is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

The x-intercept of this tangent is $(x_2, 0)$. Now, letting $y = 0$ and $x = x_2$ we get

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

Solving for x_2 yields

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

The next approximation follows the same algorithm, so the nth approximation becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This, then is the equation we use to generate successive approximations of the zero.

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton's method problems are calculator driven. To perform Newton's method on your TI-89 follow these steps:

- Set the desired number of decimal places in MODE.
- Enter the function in y1.
- Put the derivative in y2: y2=d(y1(x),x)
- Return to the home screen
- In the entry line put the Newton's method expression: x-((y1(x)/y2(x))—x=x_1 (The x_1 is your initial guess.)
- Instead of pressing ENTER, use GREEN KEY then ENTER to get a decimal approximation–this is your x_2–your second approximations of the zero.
- Now, just erase your initial guess and enter ANS(1)
- Now if you continue to press GREEN KEY and ENTER you will get successive approximations of the zero.
- Continue pressing GREEN KEY and ENTER until you get a result the repeats itself–this is your approximation of the zero.

Example 1

Find the real root of $x^3 - 4x^2 - 2 = 0$ to four decimal places. Use $x_1 = 4.5$

Set the MODE to FIX 4
Make y1=$x^3 - 4x^2 - 2$
Make y2=d(y1(x),x)
Go HOME
Enter x-((y1(x)/y2(x))|x=4.5
GREEN KEY ENTER–this is your second approximation, x_2 and should be 4.1717
Backspace over the 4.5 and enter ANS(1) using 2nd and (-)
GREEN KEY ENTER–this is your third approximation and should be 4.1192
Hit GREEN KEY ENTER again and get 4.1179
Hit GREEN KEY ENTER again and get 4.1179

You now have a four decimal place approximation of the root . . . 4.1179

Example 2

Using Newton's method, find $\sqrt{3}$ to five decimal places.

To do this, find the solution to the equation $x^2 - 3 = 0$ and use Newton's method to find the zero.

Example 3

Use Newton's method to find the intersection of $y = \frac{1}{2}x$ and $y = \sin x$ to six decimal places.

To do this, find the solution to the equation $\frac{1}{2}x = \sin x \longrightarrow \frac{1}{2}x - \sin x = 0$ and use Newton's method to find the zero.

Chapter 5

Logarithmic and Exponential Functions

5.1 Inverse Functions

5.1.1 Introduction

Inverses are pairs of functions in which the input and output have been reversed. They are essentially pairs of functions that "undo" each other. If f is the function given by $(1, 2), (3, 4), (5, 6)$, then its inverse would be $(2, 1), (4, 3), (6, 5)$. Note that the x and y values have been switched–the domain of f becomes the range of its inverse and the range of f becomes the domain of its inverse. This fact will allow us to more easily find the range of some functions–which we have not always been able to do.

The statement that inverses "undo" each other can be expressed by the following statement:

> If f and g are inverses, then
>
> $f(g(x)) = x$ and $g(f(x)) = x$

This gives us a process by which we can prove (or disprove) that two functions are inverses.

Example 1

Show that $f(x) = x + 2$ and $g(x) = x - 2$ are inverses.

Since $f(g(x)) = f(x - 2) = (x - 2) + 2 = x$

and $g(f(x)) = g(x + 2) = (x + 2) - 2 = x$

f and g are inverses.

One important item to keep in mind is that we are studying inverse *functions*. This means that the inverse muse be a function in order to be called an inverse. In previous courses you may have found "inverses" of any function and called them inverses. Take for instance the function $f(x) = x^2$. You probably "switched the x and y and solved for y. This would yield something like the following:

$$y = x^2$$

$$x = y^2$$

$$y = \pm\sqrt{x}$$

The problem here is that $y = \pm\sqrt{x}$ is definitely not a function! In fact, $f(x) = x^2$ has no inverse.

One more bit in notation before we move on. The inverse of f is denoted f^{-1} and is read "f inverse". Using this notation we have the following statement:

$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

5.1.2 Determining if a function has an inverse

When the input and output are reversed, the graph of the original function is reflected across the line $y = x$. Once the original is reflected, the resulting graph must be a function and thus must pass a vertical line test. If a vertical line is reflected across the line $y = x$, it becomes a horizontal line. This means that the original function must pass a horizontal line test if its reflection across $y = x$ is to pass a vertical line test.

The problem here is that a graph or picture is almost never a sufficient justification. We need a better, more rigorous way to show that a function has or does not have an inverse.

Functions that would pass a horizontal line test have a special characteristic. Any particular function value can only occur for a specific input. For the function $f(x) = x^2$, the function value 4 can be obtained by using a 2 or a -2 as an input. This is why $f(x) = x^2$ fails the horizontal line test. Functions that have outputs paired with one and only one input are called "monotonic" or "one-to-one" functions. So, the only functions that will have inverses are functions that are monotonic. That's great but now how do we show that a function is monotonic?

Let's look at $f(x) = x^2$ again. The reason that it is not monotonic is that the graph of the function decreases and then increases…causing a horizontal line to cross it twice. In order to avoid having a horizontal line cross the graph of a function, the function must either always increase or always decrease! That being said, we have a wonderful tool for determining if a function is increasing or decreasing…the derivative!

Functions that always increase or always decrease are monotonic. So, if $f'(x) \geq 0\ \forall x$ or if $f'(x) \leq 0\ \forall x$, f will be monotonic and therefore will have an inverse.

If $f'(x) \geq 0\ \forall x$ or if $f'(x) \leq 0\ \forall x$ then f is monotonic and will have an inverse.

Example 2

Determine if $f(x) = x^2$ has an inverse.

$$f'(x) = 2x$$

Since $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, f is not monotonic and does not have an inverse.

Example 3

Determine if $f(x) = \sqrt[3]{x}$ has an inverse.

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

Since $f'(x) > 0$ on $(-\infty, 0) \cup (0, \infty)$, f is monotonic and does have an inverse.

It is find for f' to fail to exist at $x = 0$. There is just a vertical tangent there. A derivative can pass through a zero slope or non-existent slope and still be considered monotonic.

5.1.3 Finding an inverse of a function

In previous courses you have always found an inverse by "switching the x and y" and solving for y. The problem here is that there is really no such operation as "switch the x and y" so we need a method to avoid the mystical process. The answer is in proper notation. If two functions are inverses then

$$y = f^{-1}(x) \text{ and } x = f^{-1}(y).$$

Using this notation allows us to "switch the x and y"without any magic.

To find an inverse, follow these steps: (example shown)

- Let $y = f(x)$. $y = x^3$
- Solve for x. $x = \sqrt[3]{y}$
- Substitute $f^{-1}(y)$ for x (since they are the same). $f^{-1}(y) = \sqrt[3]{y}$
- You now have a normal function $f^{-1}(y)$ so you can find any function value you want…like $f^{-1}(4)$ or $f^{-1}(8)$
- Now just find $f^{-1}(x)$. $f^{-1}(x) = \sqrt[3]{x}$

Example 4

Given $f(x) = 7x - 5$, find $f^{-1}(x)$.

Let $y = f(x)$

$$y = 7x + 5$$

$$x = \frac{y - 5}{7}$$

Since $x = f^{-1}(y)$

$$f^{-1}(y) = \frac{y - 5}{7}$$

$$f^{-1}(x) = \frac{x-5}{7}$$

See, no magic!!

Example 5

Given $f(x) = \frac{x-2}{x+2}$, find $f^{-1}(x)$

$$f(x) = \frac{x-2}{x+2}$$
$$y = \frac{x-2}{x+2}$$
$$y(x+2) = x-2$$
$$xy + 2y = x - 2$$
$$xy - x = -2y - 2$$
$$x(y-1) = -2y - 2$$
$$x = \frac{-2y-2}{y-1}$$
$$f^{-1}(y) = \frac{-2y-2}{y-1}$$
$$f^{-1}(x) = \frac{-2x-2}{x-1}$$

5.1.4 Domain and range of inverses

The ordered pairs of inverse functions are reversed. This means that:

The domain of f is the range of f^{-1}

The range of f is the domain of f^{-1}

The domain of f^{-1} is the range of f

The range of f^{-1} is the domain of f

5.1.5 Graphing an inverse

To graph the inverse of a function, simple reflect it across the line $y = x$.

5.1.6 Can we find all inverses?

The Fundamental Theorem of Algebra basically tells us that an nth degree equation has n solutions. It does not say that we can always find them, it just tells us how many solutions there are. In fact, it has also been proven that not all equations of 5th degree or higher can be solved by algebraic methods. This leads us to a problem of sorts.

Consider the function $f(x) = x^5 + 5x^3 + 2x + 3$. It's derivative is $f'(x) = 5x^4 + 15x^2 + 2$. Since $f'(x) > 0\ \forall x$, we know f is monotonic and therefore has an inverse. Let's try to find it. First we write $y = x^5 + 5x^3 + 2x + 3$. Now we have to solve for x... but we can't. It just cannot be done using algebraic techniques. This means that we know that the function has an inverse but we can never find it! Does this mean we can never know anything about the inverse? Of course not. If we couldn't, our lesson would be over... but its not! In fact, we can determine the slope of a tangent to the inverse without ever actually finding the inverse–much less the derivative of the inverse. It's really sort of crazy. We can write equations of tangents to an inverse at any x value we want–without ever finding the inverse or it's derivative! Let's see how.

By the way, the answer to the question in the section title... no we cannot always find an inverse... even if we know one exists.

5.1.7 The derivative of an inverse

There are situations when it is relatively easy to find the slope of a tangent to an inverse. This will happen when we can actually find the inverse itself and then it's derivative.

Example 6

Given $f(x) = x^3$, find $\left(f^{-1}\right)'(8)$.

We know how to find an inverse so we'll get right to the point...

If $f(x) = x^3$, then $\left(f^{-1}\right)'(x)\sqrt[3]{x}$

Now, $\left(f^{-1}\right)'(8) = \dfrac{1}{12}$.

That one was easy. We could actually find the inverse, find its derivative and find the value of the derivative at $x = 8$.

Can we find the slope of the tangent to the inverse without finding the inverse in the first place? Yes.

Consider a function f and an equation of a tangent to f at some point (c, d). The equation of the tangent is

$$y - d = f'(c)(x - c)$$

If this tangent line is reflected across the line $y = x$, we will have a tangent to the inverse of f at the point (d, c) so let's find the inverse of this tangent line.

$$x = \frac{y - d}{f'(c)} + c$$

$$f^{-1}(y) = \frac{y-d}{f'(c)} + c$$

$$f^{-1}(x) = \frac{x-d}{f'(c)} + c$$

$$f^{-1}(x) - c = \frac{1}{f'(c)}(x-d)$$

The last line is the equation of the tangent to f^{-1} at the point (d, c). Notice that the slope of this tangent is the reciprocal of the value of the derivative of the function (*not the inverse*) at $x = c$!

We now have a way to find the value of the derivative of an inverse without finding the inverse or its derivative!

If (c, d) is on f then

$$\left(f^{-1}\right)'(d) = \frac{1}{f'(c)}$$

Example 7

Given $f(x) = x^5 - x^3 + 2x$, find $\left(f^{-1}\right)'(2)$.

Begin by finding out what value of x will yield a 2. In other words, solve $f(x) = 2$. This will give us the point on f that we need.

$$x^5 - x^3 + 2x = 2 \longrightarrow x = 1$$

We now know that the point $(2, 1)$ is on f.

Since $(2, 1)$ is on f, $\left(f^{-1}\right)'(2) = \dfrac{1}{f'(1)}$

$$f'(x) = 5x^4 - 3x^2 + 2 \longrightarrow f'(1) = 4$$

$$\therefore \left(f^{-1}\right)'(2) = \frac{1}{4}.$$

Amazing! We just found the slope of a tangent to the inverse of f without ever finding f^{-1} or its derivative!

5.2 Exponential Functions and their Derivatives

5.2.1 Introduction

Exponential functions are functions of the form $f(x) = a^x$ where a is a positive constant. Some examples would be $f(x) = 7^x$, $f(x) = 2^x$, and most importantly for us, $f(x) = e^x$. Power functions have a variable raised to a positive constant, like $f(x) = x^5$. Exponential functions have a positive constant raised to a function.

One of the many differences between power and exponential functions is in their rates of growth...how quickly their function values increase. Exponential functions will increase at rate tremendously faster than a power function.

Consider the two functions $g(x) = x^2$ and $f(x) = 2^x$. The table below shows some selected function values.

x	$g(x)$	$f(x)$
0	0	1
1	1	2
2	4	16
5	25	32
10	100	1,024
20	400	1,048,576
30	900	1,073,741,824

As you can see, the exponential function values increase at a much faster rate.

Fun little problem...
If 500 sheets of paper are 2 inches thick and if you folded a piece of paper in half 50 times, how thick would it be? (I know you can't fold it more than 7 or so times but hey, this is math class and anything's possible!)

5.2.2 Properties of exponentials

The first thing you should notice is that all functions of the form $f(x) = a^x$ pass through the point $(0, 1)$. This is because $a^0 = 1$.

Here are some other properties:

$$a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y}$$

$$(a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

$$\text{If } a > 0 \text{ then } \lim_{x\to\infty} a^x = \infty \text{ and } \lim_{x\to-\infty} a^x = 0$$

$$\text{If } a < 0 \text{ then } \lim_{x\to\infty} a^x = 0 \text{ and } \lim_{x\to-\infty} a^x = \infty$$

5.2.3 The derivative of exponential functions

Here's the plan. We will first attempt to find the derivative of $f(x) = a^x$, the general exponential function. I'll warn you, we're going to hit a snag. Once we get "stuck" we'll switch gears and look at the function $f(x) = e^x$, which is every mathematician's favorite exponential function. A little investigation of a special limit involving e^x and we'll have the derivative of $f(x) = e^x$ and then later on we'll go after the derivative of $f(x) = a^x$.

If $f(x) = a^x$, then by the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x(a^h-1)}{h} \\ &= a^x \ \lim_{h\to 0} \frac{a^h-1}{h} \end{aligned}$$

Now, note that if $f(x) = a^x$ and we were looking for $f'(0)$, by definition we would have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{a^{0+h}-a^0}{h}$$

$$= \lim_{h\to 0} \frac{a^h-1}{h}$$

This is the same limit that is in the last step of our attempt to find $f'(x)$ above.

Substituting $\lim_{h\to 0} \frac{a^h-1}{h}$ into the expression $a^x \ \lim_{h\to 0} \frac{a^h-1}{h}$ we get

$$f'(x) = a^x f'(0).$$

Now this is both troublesome and interesting. Troublesome because it appears that in order to find the derivative of $f(x) = x^x$ we first need to find $f'(0)$... which we cannot because we are trying to find the derivative in the first place! It is interesting because we have discovered a very important fact. Since $f'(0)$ is always a constant, we now know that the derivative of an exponential function is always the function itself times some constant!

We've actually gone as far as we can right now with the derivative of the general exponential function, a^x. We now move on to a particular exponential function, everyone's favorite, $f(x) = e^x$.

If we had started with trying to find the derivative of $f(x) = e^x$, we would have ended at the step where we had

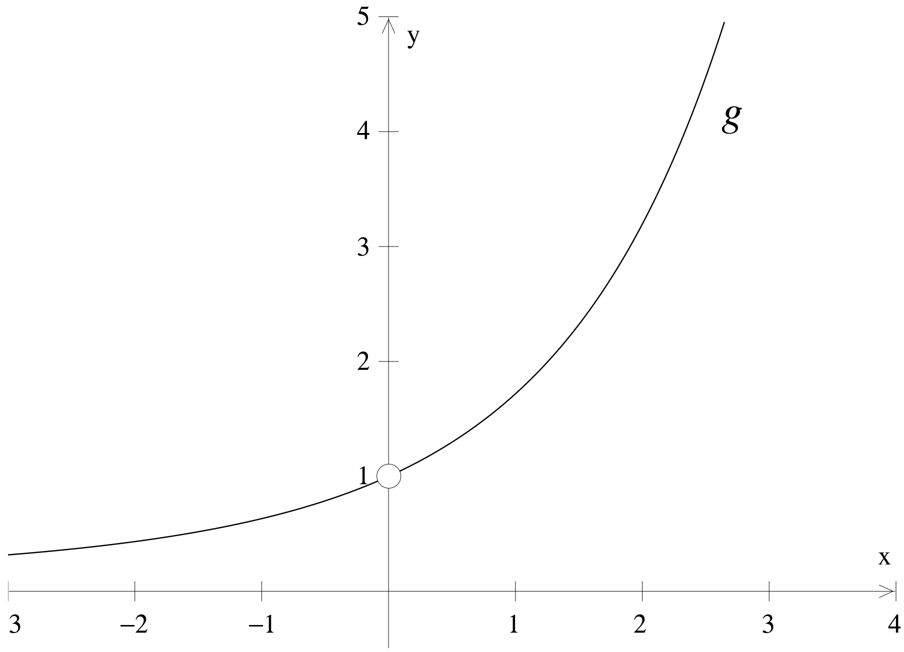
$$a^x \ \lim_{h\to 0} \frac{a^h-1}{h}$$

and for $f(x) = e^x$ this step would become

$$f'(x) = \lim_{h\to 0} \frac{e^h-1}{h}$$

Let's take a look at this limit. We know this is $f'(0)$ but we'd like to know its value. Unfortunately, actually finding the exact value of this limit is beyond the scope of this course so we will have to take a leap of faith and explore it graphically.

Below is the graph of $g(x) = \dfrac{e^x - 1}{x}$.



Even though g clearly does not exist at $x = 0$, we can still see that the limit of g as $x \to 0$ is 1.

Since this limit is 1, then if $f(x) = e^x$ then $f'(x) = e^x$. We now know the derivative of at least one exponential function, the natural exponential function.

$$\text{If } f'(x) = e^x \;\longrightarrow\; f'(x) = e^x$$

And, yes, it's everyone's favorite derivative!

We now apply the chain rule.

$$\text{If } h'(x) = e^{f(x)} \;\longrightarrow\; h'(x) = e^{f(x)} f'(x)$$

or

$$D_x \left[e^{f(x)}\right] = e^{f(x)} f'(x)$$

When we actually take the derivative of $e^{f(x)}$, we will often put the "$f'(x)$" out in front of the $e^{f(x)}$ as you will see in the following examples.

Example 1

Given $f(x) = e^{7x}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \left(e^{7x}\right) 7 \\ &= 7e^{7x} \end{aligned}$$

The first step is often skipped and the answer is usually just immediately written as $f'(x) = 7e^{7x}$

Example 2

Given $f(x) = e^{\sin 8x}$. find $f'(x)$.

$$\begin{aligned} f'(x) &= e^{\sin 8x}(\cos 8x)(8) \\ &= 8e^{\sin 8x} \cos 8x \end{aligned}$$

Example 3

Given $f(x) = xe^x$, find $f'(x)$.

$$\begin{aligned} f'(x) &= (x)\left(e^x\right) + \left(e^x\right)(1) \\ &= xe^x + e^x \\ &= e^x(x+1) \end{aligned}$$

Example 4

Given $f(x) = xe^{-x^2}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= (x)\left(-2xe^{-x^2}\right) + \left(e^{-x^2}\right)(1) \\ &= -2x^2e^{-s^2} + e^{-x^2} \\ &= e^{-x^2}\left(1 - 2x^2\right) \end{aligned}$$

Example 5

Find an equation of a tangent to $2e^{xy} = x + y$ at $(0, 2)$.

We already have the point, so all we need to do is implicitly differentiate to find the slope.

$$2e^{xy} = x + y$$

$$2e^{xy}\left(x\frac{dy}{dx} + y\right) = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1 - 2ye^{xy}}{2xe^{xy} - 1}$$

$$\left.\frac{dy}{dx}\right|_{(0,2)} = 3 \ \therefore m_T = 3$$

The equation of the tangent is $y - 2 = 3(x - 0)$.

5.2.4 Limits at infinity involving the natural logarithmic function

Many texts treat these limits differently than I will. I suggest you use your common sense and play the usual "big number game". Always begin by eliminating any negative exponents and getting a common denominator.

Example 6

Evaluate: $\lim\limits_{x\to\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} &= \lim_{x\to\infty} \frac{e^{3x} - \frac{1}{e^{3x}}}{e^{3x} + \frac{1}{e^{3x}}} \\
&= \lim_{x\to\infty} \frac{e^{3x} - \frac{1}{e^{3x}}}{e^{3x} + \frac{1}{e^{3x}}} \cdot \frac{e^{3x}}{e^{3x}} \\
&= \lim_{x\to\infty} \frac{e^{6x} - 1}{e^{6x} + 1}
\end{aligned}$$

Now, the 1 and the -1 do not matter. We have a "really big number" over the same "really big number" so we get

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = 1$$

5.3 Logarithmic Functions and their Derivatives

5.3.1 Introduction

Logarithmic functions are functions of the form $f(x) = \log_a x$ or $y = \log_a x$. The logarithm of a number x is simply the exponent to which the base a must be raised to obtain x. For example, $\log_2 8 = 3$ because $2^3 = 8$. This fact leads us to the following definition.

$$\log_a x = y \Leftrightarrow a^y = x$$

Example 1

Find $\log_3 81$.

$$\begin{aligned}\log_3 81 &= x\\ 3^x &= 81\\ x &= 4\end{aligned}$$

Example 2

Solve $\log_{25} 5 = x$.

$$\begin{aligned}\log_{25} 5 &= x\\ 25^x &= 5\\ \left(5^2\right)^x &= 5\\ 5^{2x} &= 5^1\\ 2x &= 1\\ x &= \frac{1}{2}\end{aligned}$$

5.3.2 Properties of logarithms

You are probably familiar the these properties but here they are.

$$\begin{aligned}\log_a xy &= \log_a x + \log_a y\\ \log_a x^y &= y \log_a x\\ \log_a \frac{x}{y} &= \log_a x - \log_a y\end{aligned}$$

It was these properties which made logarithms so useful when they were invented by Napier in the first half of the 17th century. Logarithms basically allow us to change a multiplication problem into an addition problem, division into subtraction and exponentiation into multiplication. They reduce the complexity of calculations. This was a boon to astronomers of the time who, prior to logarithms, spent the majority of their time in tedious calculations rather than observation.

We now introduce two more properties of logarithms. Consider $log_a a^x$. This is basically asking, "To what must a be raised to get a^x?" The obvious answer is x! Now consider $a^{\log_a x}$. The definition of logarithms tells us that if $\log_a x = y$ then $a^y = x$ and therefore $a^{\log_a x} = x$.

$$a^{\log_a x} = x$$

and

$$\log_a a^x = x$$

The logarithmic function $f(x) = \log_a x$ has a domain of $(0, \infty)$ and a range of $(-\infty, \infty)$. It is always increasing and differentiable everywhere. As you have guessed, logarithmic and exponential functions are inverses of each other.

5.3.3 The natural exponential function

Just as the most important exponential function for us is the natural exponential function, $f(x) = e^x$, the most important logarithmic function is the natural logarithmic function, $f(x) = \log_e x$. This function is used so frequently, it has its own special notation, $f(x) = \ln x$. The properties listed above for logarithmic functions also apply to the natural logarithmic function.

$$\ln x = y \ \Leftrightarrow e^y = x$$

$$\ln e^x = x$$

$$e^{\ln x} = x$$

The last two properties are critical. You will see them both on exams in this class as well as on the actual AP test.

Just remember that e to the $\ln$ of "anything" is just the "anything" and that the natural log of e to the "anythin" is just the "anything".

There is just one more very useful property we need to discuss. Any base logarithm can be expressed in terms of any other base logarithm. Because our calculators only "know" base 10 and base e, we need to convert any other base to one of these two bases. . . most often base e.

If $y = \log_a x$

then $a^x = y$

Taking the natural log of both sides yields

$$\ln a^y = \ln x$$

$$y \ln a = \ln x$$

$$y = \frac{\ln x}{\ln a}$$

But we know $y = \log_a x$ so we get

$$\log_a x = \frac{\ln x}{\ln a}$$

5.3.4 Derivatives of logarithmic functions

We will first concern ourselves with the derivative of $f(x) = \ln x$. After we have that derivative, we will discuss the derivative of any base logarithmic function.

We begin with $y = \ln x$.

This implies $e^y = x$.

Implicitly differentiating yields $e^y \dfrac{dy}{dx} = 1$.

$$\frac{dy}{dx} = \frac{1}{e^y}$$

But $e^y = x$ so

$$\frac{dy}{dx} = \frac{1}{x}$$

We now apply the chain rule to get our final derivative.

$$D_x\left[\ln f(x)\right] = \frac{1}{f(x)} f'(x)$$

or

$$D_x\left[\ln f(x)\right] = \frac{f'(x)}{f(x)}$$

Example 3

Given $f(x) = \ln(\sin x)$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{1}{\sin x}\cos x \\ &= \frac{\cos x}{\sin x} \\ &= \cot x \end{aligned}$$

Normally, the first step is skipped. Just think “the derivative of the the function over the function”.

Example 4

Given $f(x) = \ln \dfrac{x+1}{\sqrt{x-2}}$, find $f'(x)$.

First of all remember to apply all the properties of logarithms *before* you take the derivative.

$$f(x) = \ln(x+1) - \ln\sqrt{x-2}$$

$$f'(x) = \frac{1}{x-1} - \frac{\frac{1}{2\sqrt{x-2}}}{\sqrt{x-2}}$$

$$f'(x) = \frac{1}{x+1} - \frac{1}{2(x-2)}$$

Example 5

Given $f(x) = \ln(\ln x)$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{\frac{1}{x}}{\ln x} \\ &= \frac{1}{x \ln x} \end{aligned}$$

5.3.5 Derivatives of general logarithmic functions

We will now derive the derivative of the general logarithmic function, $f(x) = \log_a x$. Since we know that

$$\log_a x = \frac{\ln x}{\ln a}.$$

We can say

$$D_x\left[\log_a x\right] = D_x\left[\frac{\ln x}{\ln a}\right].$$

Now, the $\frac{1}{\ln a}$ is a constant,

$$\begin{aligned} D_x\left[\log_a x\right] &= \frac{1}{\ln a} D_x\left[\ln x\right] \\ &= \frac{1}{\ln a}\frac{1}{x} \\ &= \frac{1}{x \ln a} \end{aligned}$$

Applying the chain rule yields the final version of our derivative.

$$D_x\left[\log_a f(x)\right] = \frac{f'(x)}{f(x)\ln a}$$

Now, the only problem with this derivative is that it changes the base. Often we want to express an answer in the same base as the original problem so we need another version of this derivative that maintains the original base.

We know that

$$\log_a x = \frac{\ln x}{\ln a}$$

If we let $x = e$ this becomes

$$\begin{aligned}\log_a e &= \frac{\ln e}{\ln a} \\ &= \frac{1}{\ln a}\end{aligned}$$

This means that whenever we see $\dfrac{1}{\ln a}$ we can replace it with $\log_a e$.

$$D_x\left[\log_a f(x)\right] = \frac{f'(x)}{f(x)\ln a} = \frac{f'(x)\log_a e}{f(x)}$$

Example 6

Given $f(x) = \log_9(2 + \sin x)$, find $f'(x)$.

$$f'(x) = \frac{\cos x}{(2 + \sin x)\ln 9}$$

or

$$f'(x) = \frac{(\cos x)\log_9 e}{2 + \sin x}$$

5.3.6 Derivative of the general exponential function

Now that we have the derivative of logarithmic functions, we can use them to help us derive the derivative of the general exponential function, $f(x) = a^x$.

We know that $a^x = e^{\ln a^x} = e^{x \ln a}$.

$$\begin{aligned}D_x\left[a^x\right] &= D_x\left[e^{x\ln a}\right] \\ &= e^{x\ln a} D_x[x\ln a]\end{aligned}$$

Now, $\ln a$ is a constant, so the derivative of $x \ln a$ is just $\ln a$.

$$D_x\left[a^x\right] = e^{x\ln a}\ln a$$

We also know that $e^{x \ln a} = e^{\ln a^x} = a^x$.

$$D_x\left[a^x\right] = a^x \ln a$$

Applying the chain rule gives us the following:

$$D_x\left[a^{f(x)}\right] = a^{f(x)} f'(x)\ln a$$

Example 7

Given $f(x) = 8^{x^2}$, find $f'(x)$.

$$f'(x) = \left(8^{x^2}\right)(2x)(\ln 8)$$

There is usually no good way to simplify these expressions so just we generally just leave them alone.

5.3.7 Logarithmic differentiation

We can now find the derivative of a function raised to a number (power rule) and the derivative of a number raised to a function. How would we address a function raised to a function. We can't use the power rule and we can treat it as an exponential function. What we can do is something called "logarithmic differentiation". We don't use this technique very often but it does come in handy at times…especially with functions raised to functions. It can also be used to simplify the differentiation of any complicated function. The process involves taking the natural log of both sides first, then simplifying using properties of logarithms and then implicitly differentiating.

Let's consider $f(x) = x^{x^3}$ and use logarithmic differentiation.

First, let $y = f(x)$.

$$y = x^{x^3}$$

Now, take the natural log of both sides.

$$\ln y = \ln\left(x^{x^3}\right)$$

Use the properties of logarithms.

$$\ln y = x^3 \ln x$$

Implicitly differentiate and don't forget to use the power rule.

$$\frac{1}{y}\frac{dy}{dx} = \left(x^3\right)\frac{1}{x} + (\ln x)\left(3x^2\right)$$

Now solve for $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = y(x^2 + 3x^2\ln x)$$

Finally substitute x^{x^3} for y.

$$\frac{dy}{dx} = x^{x^3}(x^2 + 3x^2\ln x)$$

Example 8

Given $f(x) = x^{\sqrt{x}}$, find $f'(x)$.

$$\begin{aligned}
f(x) &= x^{\sqrt{x}} \\
y &= x^{\sqrt{x}} \\
\ln y &= \ln x^{\sqrt{x}} \\
\ln y &= \sqrt{x} \ln x \\
\frac{1}{y} \frac{dy}{dx} &= \sqrt{x} \, \frac{1}{x} + (\ln x) \, \frac{1}{2\sqrt{x}} \\
\frac{dy}{dx} &= y \left(\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}} \right) \\
\frac{dy}{dx} &= x^{\sqrt{x}} \left(\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}} \right)
\end{aligned}$$

5.4 Exponential Growth

5.4.1 Introduction

Exponential growth is an example of direct variation. Direct variation is a relationship in which the dependent variable varies as a factor of the independent variable. Direct variation is often expressed

$$y = k \; x.$$

In exponential growth, the rate of growth in a quantity is directly proportional to the amount present at some initial time. This is expressed as

$$\frac{dy}{dt} = k \; y.$$

where y is the amount present, dy/dx is the rate of growth of y with respect to time t, and k is the growth constant. As a matter of fact, this relationship is the very definition of exponential growth.

<u>Exponential Growth</u>

$$\frac{dy}{dt} = k \; y$$

What we want to do now come up with a model for exponential growth that does not involve any derivatives. The definition stated above is an example of a "differential equation"–and equation with a derivative in it. We will spend a good deal of time solving simple differential equations later on in the course. For now, we will try to find a different way.

Let's start with the definition.

$$\frac{dy}{dt} = k \; y$$

We now divide both sides by y.

$$\frac{1}{y}\frac{dy}{dt} = k$$

OK... now we need a function such that if we differentiated it with respect to t, we would get $\frac{1}{y}\frac{dy}{dt}$. As you've probably guessed, it's $\ln y$.

On the right side we have a k. We need a function in t whose derivative with respect to t is k. This, of course, is kt.

This process of "backwards differentiating"... looking for a function whose derivative is given is called "antidifferentiation" and we'll spend a huge amount of time on it.

Before we move on, we need to talk about one more facet of this antidifferentiation. If we wanted a function whose derivative was $2x$, we would choose x^2 because the derivative of x^2 is $2x$. The problem is that there are a lot of functions whose derivative is $2x$. Here are just a few: x^2, $x^2 + 4$, $x^2 - 7$, etc. Notice that they all differ by a constant. To account for this, when we antidifferentiate we always include a constant. So the real antiderivative of $2x$ is $x^2 + C$.

Since we have antidifferentiated, we should have a constant on both sides of our equation–but we'll just lump them together on the right side.

Back to our problem.

$$\frac{1}{y}\frac{dy}{dt} = k$$
$$\ln y = kt + C$$
$$y = e^{kt+C}$$
$$y = e^{kt}e^{C}$$

But since e^C is just a constant we get

$$y = A\ e^{k}t.$$

Now, when $t = 0$ the amount present is the original amount which we call y_0. Substituting these values yields

$$y_0 = A\ e^0.$$

This means that the $A = y_0$ is the original amount present. This all results in our exponential growth model.

<u>Exponential Growth Model</u>

$$y = y_0 e^{kt}$$

When you use this model, y is the amount present after time t. y_0 is the original amount. k is the growth constant and t is time.

When this model is applied to problems involving money you will see the following form.

$$A = Pe^{rt}$$

In this model, A is the amount of money after t time, P is the original amount invested, r is the interest rate and t is time.

5.4.2 Types of exponential growth problems

We normally face two types of problems involving exponential growth.

In the easier of the two types, we are given a data point–usually a time and an amount present–and the growth rate (k). These problems can be done by simply substituting the data into our model and solving.

In the second type of problem, we are given two data points. In this case, our first task is to find the growth constant k. Once we have an initial amount and the growth constant, we can make the model specific to our problem and solve for either a time or an amount.

Example 1

In a certain culture, the rate of growth of bacteria is directly proportional to the amount present. If 1000 bacteria are initially present and the amount doubles in 12 minutes, how long will it be before there are one million bacteria?

Our first step is to pull out the two data points and use them to find the growth constant.

When $t = 0,\ y = 1000$ and when $t = 12,\ y = 2000$. Our data points are $(0, 1000)$ and $(12, 2000)$.

Substituting these values into our model yields

$$2000 = 1000e^{12k} \longrightarrow k = \frac{\ln 2}{12}.$$

Our model now specified to our problem is

$$y = 1000e^{\frac{\ln 2}{12}t}$$

Now, let $y = 1000000$ and solve for t.

$$1000000 = 1000e^{\frac{\ln 2}{12}t} \longrightarrow t \approx 119.589$$

Therefore there will be one million bacteria present after 119.589 minutes.

Note: When you get an expression for k, store it in your calculator. Do not round it off to three decimal places and then store it! This will cause rounding errors.

Example 2

The rate of increase in the population of a certain city is directly proportional to the population. If the population in 1950 was 50,000 and in 1980 it was 75,000, what will the population be in 2010?

First of all, years are not real numbers, they are labels. We will consider 1950 as $t = 0$ and so 1980 will be $t = 30$ and 2010 will be $t = 60$.

Our data points are $(0, 50000)$ and $(30, 75000)$.

Substituting into the model yields

$$75000 = 50000e^{30k} \longrightarrow k = \frac{1}{30}\ln\frac{3}{2}.$$

Now our model becomes

$$y = 50000e^{\frac{1}{30}\ln\frac{3}{2}\,t}$$

Now let $t = 60 \longrightarrow y = 112500$

Therefore in 2010 there will be 112,500 people.

Example 3

The rate of decay of radium is directly proportional to the amount present at any time. If 60 mg. are present now and its half-life is 1690 years, how much radium will be present 100 years from now?

Decay problems are done the same way as growth problems. The only difference is that you will have a negative growth constant.

The two data points are $(0, 60)$ and $(1690, 30)$.

$$30 = 60e^{1690k} \longrightarrow k = -\frac{\ln 2}{1690}$$

Our model now becomes

$$y = 60e^{-\frac{\ln 2}{1690}\,t}$$

Now let $t = 100$ and solve for y.

$$y = 60e^{-\frac{\ln 2}{1690}\,100} \longrightarrow y \approx 57.589$$

Therefore after 100 years there will be about 57.589 mg of radium.

Note: there is actually a much quicker way to do this problem which we will see in the next section.

5.4.3 Half-life and doubling time

We can easily derive a simple formula to determine the half-life or doubling time of a given quantity undergoing exponential growth or decay.

Consider our exponential growth model where the original amount is twice the amount present. In this case, $y_0 = 2y$.

$$y = y_0e^{kt}$$

$$1 = 2e^{kt}$$

Solving for t yields

$$t = -\frac{\ln 2}{k}$$

If we know the half-life, we can easily find the growth constant. If we know the growth constant, we can easily find the half-life. In Example 3, we knew the half-life was 1690 years so we could have just solved

$$1690 = -\frac{\ln 2}{k}$$

to find the k.

<u>Half-life</u>

$$t = -\frac{\ln 2}{k}$$

Doubling time (mostly used in problems involving money) is derived the same way with the exception that the k will be positive.

<u>Doubling time</u>

$$t = \frac{\ln 2}{k}$$

5.5 The Inverse Trigonometric Functions

5.5.1 Introduction

We would be remiss if we had an entire discussion of inverse functions and left out the six trigonometric functions. The problem is, none of them are monotonic and thus cannot have inverses. No inverse trig functions? We can't have that so we do what any good mathematician would do... change the problem! We can "fix" this problem by restricting the domains of the trigonometric functions so that they are monotonic.

By the way, you will see two different notations for the inverse trigonometric functions. Inverse sine, for instance, can be written as $\sin^{-1} x$ or as $\arcsin x$. The "arc" notation actually gets closer to the notion of how we measure angles in mathematics. As you know, we measure angles in radians. A radian is a distance... an arc length. When we measure angles in mathematics we are measuring the length of arc the angle subtends. Thus, if we are asked to find $\arcsin 1$, we are being asked to find the angle whose sine is 1... find the angle whose arc length is 1. The notation with the -1 seems to be more widely used these days. It does match up nicely with our usual inverse notation, f^{-1}.

5.5.2 Inverse sine

To get the inverse sine function, we restrict the domain of $\sin x$ and create a "new" sine function which is monotonic. We restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The range remains $[-1, 1]$.

This means that the domain of $\sin^{-1} x$ is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Now, since the range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the answer to inverse sine questions must be in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

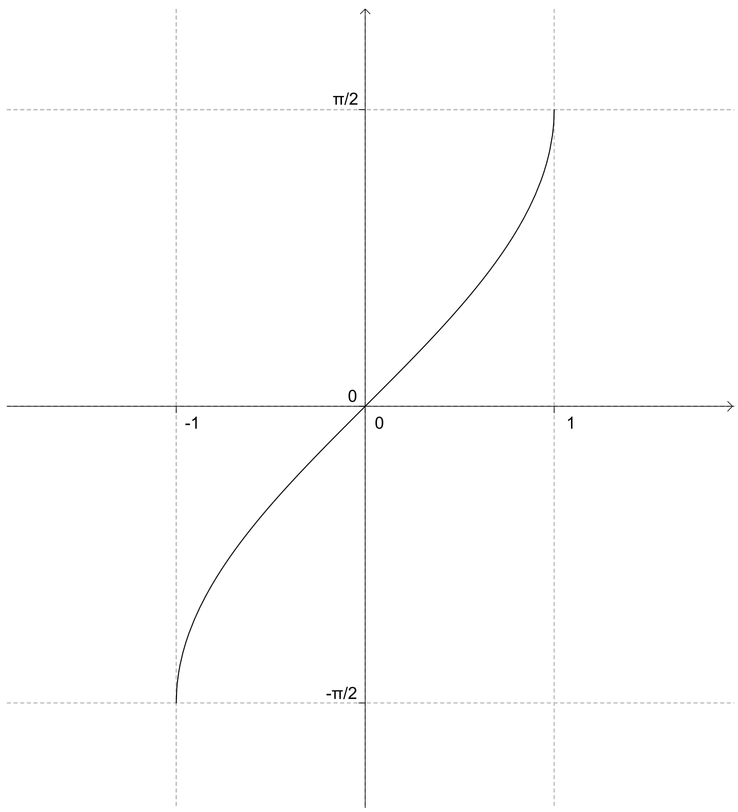
<u>Inverse Sine</u>

$$y = \sin^{-1} x \text{ iff } x = \sin x \text{ where } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sin\left(\sin^{-1}\right) = x \text{ for } x \in [-1, 1]$$

$$\sin^{-1}(\sin x) = x \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Below is the graph of inverse sine.



Example 1

Find $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

First, draw the appropriate reference angles. They should be in the first and second quadrant. Since the range of inverse sine is $(-\pi/2, \pi/2)$, the angle in the first quadrant is used.

$$\text{Thus } \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

Example 2

Find $\sin^{-1}\left(\sin\frac{\pi}{12}\right)$.

$$\text{Since } \frac{\pi}{12} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \sin^{-1}\left(\sin\frac{\pi}{12}\right) = \frac{\pi}{12}.$$

Example 3

Find $\sin\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$.

$$\text{Since } \frac{1}{\sqrt{2}} \in [-1, 1],\ \sin\left(\sin^{-1}\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

Example 4

Find $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$.

Since $\frac{7\pi}{6} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have to first find $\sin\frac{7\pi}{6}$ using reference angles or the unit circle.

We then have $\sin^{-1}\left(-\frac{1}{2}\right)$.

Now draw two reference angles and use the one in the fourth quadrant.

$$\text{Therefore } \sin^{-1}\left(\sin\frac{7\pi}{6}\right) = -\frac{\pi}{6}$$

5.5.3 Inverse cosine

To get the inverse cosine function, we restrict the domain of $\cos x$ and create a "new" cosine function which is monotonic. We restrict the domain to $[0, \pi]$. The range remains $[-1, 1]$.

This means that the domain of $\cos^{-1} x$ is $[-1, 1]$ and the range is $[0, \pi]$.

Now, since the range of $\cos^{-1} x$ is $[0, \pi]$ the answer to inverse sine questions must be in $[0, \pi]$.

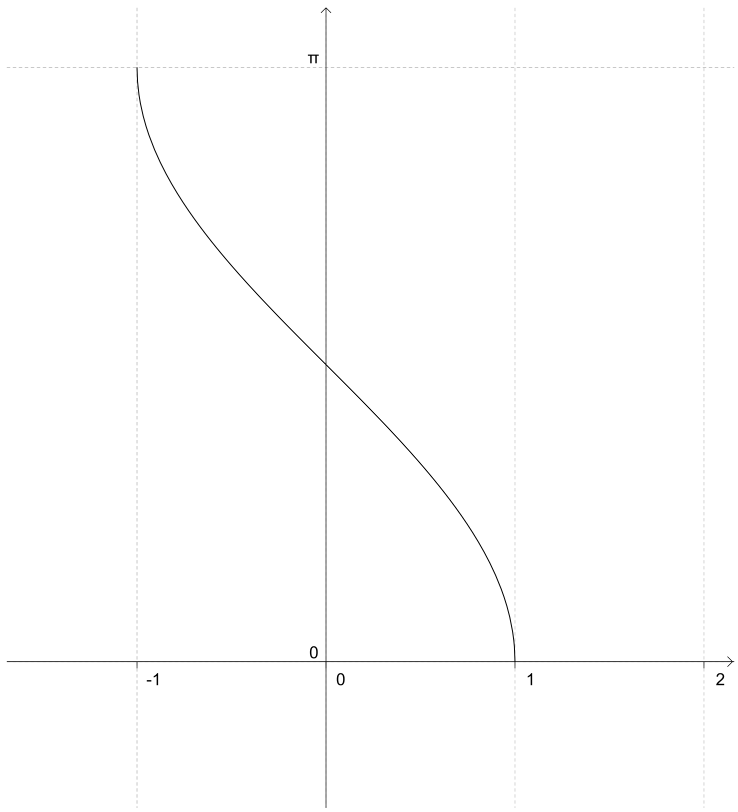
Inverse Cosine

$$y = \cos^{-1} x \text{ iff } x = \cos x \text{ where } y \in [0, \pi]$$

$$\cos\left(\cos^{-1}\right) = x \text{ for } x \in [-1, 1]$$

$$\cos^{-1}(\cos x) = x \text{ for } x \in [0, \pi]$$

Below is the graph of inverse cosine.



Example 5

Find $\cos^{-1} 0$.

We are simply being asked to find an angle in $[0, \pi]$ whose cosine is zero.

Using the unit circle we determine that $\cos^{-1} 0 = \dfrac{\pi}{2}$.

Example 6

Find $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$.

Draw the two reference angles. They will be in the second and third quadrant. The one we need must be in the second quadrant.

$$\text{Therefore } \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

5.5.4 Inverse tangent

To get the inverse tangent function, we restrict the domain of $\tan x$ and create a "new" tangent function which is monotonic. The restriction is very similar to the restriction we put on sine. We restrict the domain to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The range remains $(-\infty, \infty)$.

This means that the domain of $\tan^{-1} x$ is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, since the range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the answer to inverse tangent questions must be in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

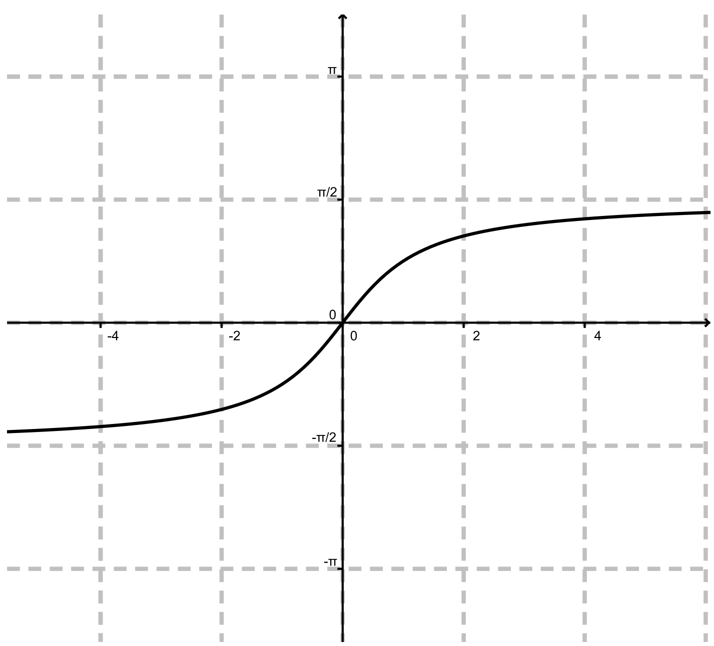
<u>Inverse Tangent</u>

$$y = \tan^{-1} x \text{ iff } x = \tan x \text{ where } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\tan\left(\tan^{-1}\right) = x \text{ for } x \in (-\infty, \infty)$$

$$\tan^{-1}(\tan x) = x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Below is the graph of inverse tangent.



Example 7

Find $\tan^{-1}\sqrt{3}$.

Since the tangent of an angle is the opposite over the adjacent, we draw the reference angles, one with an opposite side of length $\sqrt{3}$ and an adjacent side of length 1 and the other with an opposite side of length $-\sqrt{3}$ and adjacent side of length -1. The first angle is in the first quadrant and the second is in the third quadrant. Since our angle must be in $(\pi/2, \pi/2)$, we choose the angle in the first quadrant.

$$\text{Therefore } \tan^{-1}\sqrt{3} = \frac{\pi}{3}.$$

Example 8

Find $\tan^{-1} 0$.

We are being asked to find an angle in $(\pi/2, \pi/2)$ whose tangent is zero. Using the unit circle and the fact that tangent is sine over cosine we determine,

$$\tan^{-1} 0 = 0.$$

5.5.5 Inverse secant, cosecant and cotangent

To be honest, we rarely use these inverses at this level of mathematics. Also, although there is general agreement within mathematics of how to restrict sine, cosine and tangent to get their inverses, there is not complete agreement on how to restrict secant, cosecant or cotangent. Different textbook authors use different restrictions. For this reason, and for the purposes of this course, you should know that these inverses can exist but we will not dwell on any details.

5.6 Derivatives of the Inverse Trigonometric Functions

5.6.1 Inverse sine

We begin with the two statements which define inverse sine.

$$y = \sin^{-1} x$$

which means

$$x = \sin y$$

Implicitly differentiating with respect to x yields:

$$1 = \cos y \, \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Now, this is the derivative but we need it in terms of x, not y.

To do this we will use the identity

$$\sin^2 y + \cos^2 y = 1$$

We know $x = \sin y$ so $x^2 = \sin^2 y$

Substituting gives us

$$x^2 + \cos^2 y = 1 \text{ or } \cos y = \sqrt{1 - x^2}.$$

Substituting this into our derivative gives us

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Applying the chain rule gives us the final form of the derivative.

$$D_x \left[\sin^{-1} f(x)\right] = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

5.6.2 Inverse cosine

We begin with the two statements which define inverse cosine.

$$y = \cos^{-1} x$$

which means

$$x = \cos y$$

Implicitly differentiating with respect to x yields:

$$1 = -\sin y \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

Now, this is the derivative but again, we need it in terms of x, not y.

To do this we will use the same identity we used for inverse sine

$$\sin^2 y + \cos^2 y = 1$$

We know $x = \cos y$ so $x^2 = \cos^2 y$

Substituting gives us

$$\sin^2 y + x^2 = 1 \text{ or } \sin y = \sqrt{1 - x^2}.$$

Substituting this into our derivative gives us

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Applying the chain rule gives us the final form of the derivative.

$$D_x\left[\cos^{-1} f(x)\right] = -\frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$$

Note that the derivative of inverse cosine is simply the negative of the derivative of inverse sine!

5.6.3 Inverse tangent

The same process is used to derive the derivative of inverse tangent as was used for inverse sine and inverse cosine.

$$y = \tan^{-1} x$$

which means

$$x = \tan y$$

Implicitly differentiating with respect to x yields:

$$1 = \sec^2 y\,\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Now, this is the derivative but again, we need it in terms of x, not y.

To do this we will use another of the Pythagorean trig identities.

$$\sec^2 y = 1 + \tan^2 y$$

We know $x = \tan y$ so $x^2 = \tan^2 y$

Substituting gives us

$$\sec^2 y = 1 + x^2.$$

Substituting this into our derivative gives us

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

Applying the chain rule gives us the final form of the derivative.

$$D_x\left[\tan^{-1} f(x)\right] = \frac{f'(x)}{1 + [f(x)]^2}$$

5.6.4 Inverse secant

We begin as usual.

$$y = \sec^{-1} x$$

which means

$$x = \sec y$$

Implicitly differentiating with respect to x yields:

$$1 = \sec y \tan y \, \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

Now, this is the derivative but again, we need it in terms of x, not y.

We will have to get both $\sec y$ and $\tan y$ in terms of x.

We already know that $x = \sec y$

That gives us

$$\frac{dy}{dx} = \frac{1}{x \tan y}$$

We also know that

$$\tan^2 y = \sec^2 y - 1$$

Thus, $\tan y = \sqrt{x^2 - 1}$ Substituting x for $\sec y$ and $\sqrt{x^2 - 1}$ for $\tan y$ yields

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

Applying the chain rule gives us the final form of the derivative.

$$D_x\left[\sec^{-1} f(x)\right] = \frac{f'(x)}{x\sqrt{[f(x)]^2 - 1}}$$

5.6.5 Inverse cosecant and inverse cotangent

Now, we've seen four of the six inverse trigonometric derivatives derived, so without loss of generality, the last two are obtained in a similar fashion.

$$D_x\left[\csc^{-1} f(x)\right] = -\frac{f'(x)}{x\sqrt{\left[f(x)\right]^2 - 1}}$$

$$D_x\left[\cot^{-1} f(x)\right] = -\frac{f'(x)}{1 + \left[f(x)\right]^2}$$

Note that the derivatives of the inverse trigonometric "co-functions" are just the negative of the derivative of their respective inverse trigonometric function.

Chapter 6

Applications of the Derivative II

6.1 Introduction

This next section of material deals with more applications of the derivative. In fact, in this chapter we will learn about the classic applications of the derivative–one of the main uses of the calculus in mathematics. We will start with a discussion of the various types of maximum and minimum function values and how we find them. Next, we use the derivative to analyze functions–determining where they are increasing or decreasing, where they are concave up or down and where (if anywhere) they have points of inflection. Finally we will do some problems involving optimization of quantities.

6.1.1 Extrema

The word "extrema" is used when we want to talk about both maximum and minimum function values. For instance, a problem may ask you to find "the extrema" of a function. This means that you would need to find all of the different types of maximum and minimum function values for a function–if indeed it has any at all. Now, there are two types of extrema, absolute and relative (or local). If we want to talk about both relative maximums and relative minimums, we use the term "relative extrema" or "local extrema". If we want to talk about both absolute maximum function values and absolute minimum function values, we use the term "absolute extrema".

First, a few definitions.

f has an **absolute maximum** $f(c)$ if $f(c) \geq f(x)$ for all x in f.

f has an **absolute minimum** $f(c)$ if $f(c) \leq f(x)$ for all x in f.

An absolute minimum function value is a function value below which the function never goes. An absolute maximum function value is a function value above which the function never goes.

Consider the function $f(x) = x^2 + 1$. This is an upward opening parabola with a vertex at $(0, 1)$. The smallest function value in this case is 1. We say f has an absolute minimum function value of 1 at $x = 0$.

Note that f does not have an absolute maximum value.

The function $g(x) = -(x-2)^2 + 3$ is a downward opening parabola with a vertex at $(2, 3)$. The largest function value we can ever get is 3. We say that g has an absolute maximum function value of 3 at $x = 2$. Note that g has no absolute minimum function value.

The function $h(x) = x^3$ is an example of a function that has neither an absolute maximum nor an absolute minimum.

So, not all functions have absolute extrema... unless they are continuous functions on a closed interval. Now, that's an entirely different situation. If a function is continuous on a closed interval, it *must* have both an absolute maximum *and* an absolute minimum function value. In fact, this is a theorem... a very famous theorem.

Extreme Value Theorem

If f is continuous on the closed interval $[a, b]$, the f has an absolute maximum function value $f(c)$ for some $c \in [a, b]$ **and** an absolute minimum function value $f(d)$ for some $d \in [a, b]$.

This is an "existence" theorem. It tells us only that these absolute extrema exist, it does not tell us how to find them.

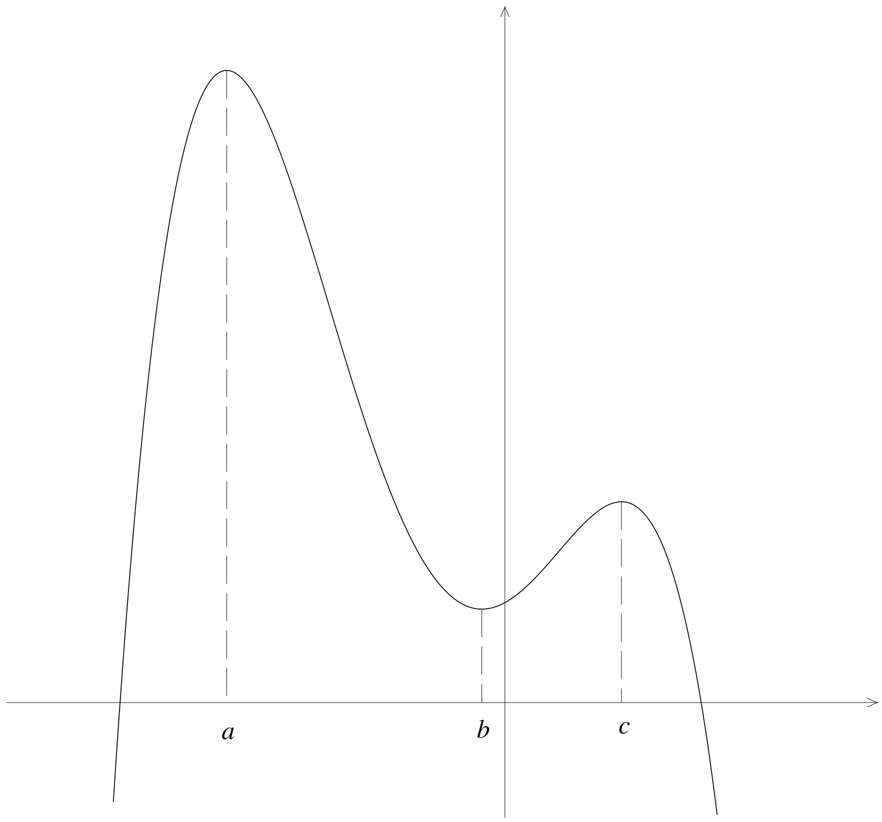
To find these absolute extrema we must first introduce another type of extrema... relative or local extrema.

f has a **relative maximum** at $x = c$ if there exists an interval I containing c such that $f(c) \geq f(x)\ \forall\ x \in I$

f has a **relative minimum** at $x = c$ if there exists an interval I containing c such that $f(c) \leq f(x)\ \forall\ x \in I$

Relative extrema must occur on an interval. We don't have relative extrema at the endpoints of intervals. You can look at it like this... if f has a a relative minimum at $x = c$ then $f(c)$ is the lowest function value in the neighborhood of $x = c$. Technically, we say that $f(c)$ is the smallest function value in the "delta-deleted neighborhood" of c.

The easiest way to get an initial idea of what relative extrema are is to look at the graph below.



The graph above has a relative maximum at $x = a$ and $x = c$. It has a relative minimum at $x = b$. Note that the relative maximum at $x = a$ is also an absolute maximum. In this case we would call $f(a)$ an absolute maximum, not a relative maximum.

Now we know what relative extrema are, but we have not yet talked about how to find them.

It's fairly plain to see that a function has a maximum when the function increases and then decreases. Similarly, a function has a minimum where the function decreases and then increases. Let's focus on maximums first. If a function is increasing, then $f'(x) > 0$. If a function is decreasing, then $f'(x) < 0$. Well, if $f'(x) > 0$ and then $f'(x) < 0$, and if f is continuous, then the Intermediate Value Theorem tells us that $f'(x) = 0$ at some point in between. In fact, the function reaches its maximum where $f'(x) = 0$. The same is true for minimums. If a function is decreasing and then increasing, the derivative goes from being negative to being positive…thus it must pass through zero.

Now, if functions have relative extrema, do they always occur where $f'(x) = 0$? Consider the graph of the function $f(x) = |x - 2|$. In this case f is decreasing before $x = 2$ and increasing after $x = 2$ but the derivative does not exist at $x = 2$. (Remember our differentiability discussion!) Derivatives do not exist at points where functions have corners or cusps.

So clearly, if a function has relative extrema those extrema must occur at places where $f'(x) = 0$ or $f'(x)$ fails to exist. In fact, it's a theorem.

Fermat's Theorem

If f has a relative extrema at $x = c$, then $f'(c) = 0$ or $f'(c) \nexists$

Values of x in the domain of f where the derivative is zero or non-existent are called **critical numbers**.

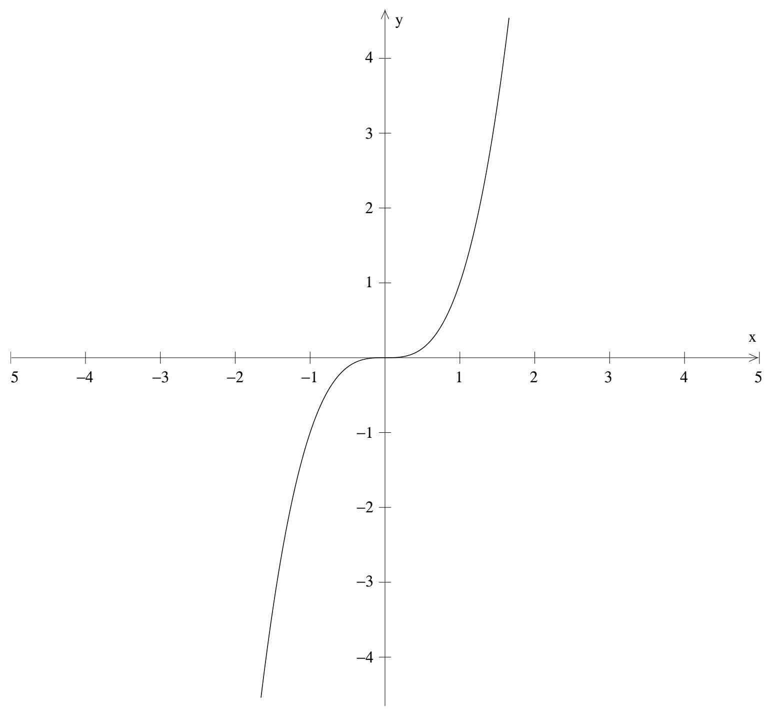
Consider $f(x) = \frac{1}{x}$.

$$f'(x) = -\frac{1}{x^2}$$

Here, $f'(x) \nexists$ at $x = 0$ but 0 is not in the domain of f so it is *not* a critical number.

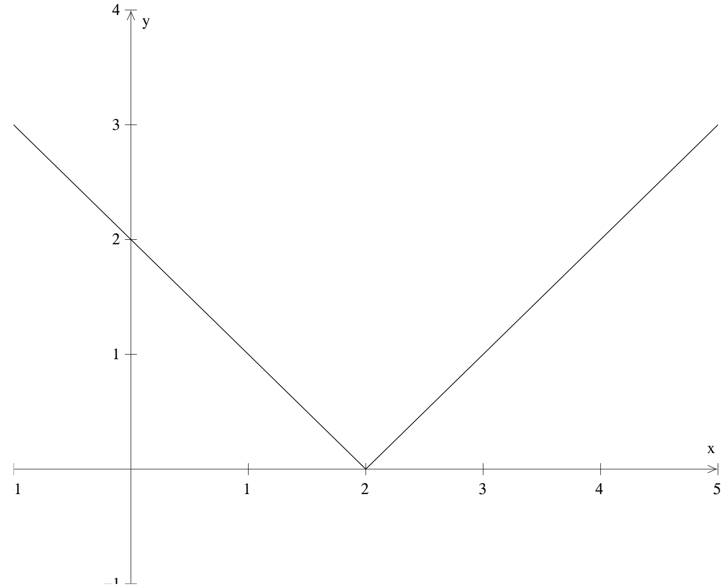
Important: Fermat's Theorem is not a two-way theorem. If a function has relative extrema, they must occur at critical numbers **but** just because a function has critical numbers *does not mean it must have relative extrema.*

Take a look at the function $f(x) = x^3$.



Since $f'(x) = 3x^2$, $f'(x) = 0$ at $x = 0$ but there is no extrema at $x = 0$. So we have a situation where the derivative is zero (a critical number) but there is no extrema.

Let's look at $f(x) = |x - 2|$.



We need to rewrite f in order to take a derivative.

$$f(x) = \begin{cases} x-2 & \text{for } x \geq 2 \\ 2-x & \text{for } x < 2 \end{cases}$$

Taking the derivative yields:

$$f'(x) = \begin{cases} 1 & \text{for } x > 2 \\ -1 & \text{for } x < 2 \end{cases}$$

Since $f'_{-}(2) = -1$ but $f'_{+}(2) = 1$, $f'(2) \nexists$.

But f clearly has a minimum at $x = 2$.

These two situations we have just explored serve to reinforce the fact that just because a derivative is zero or non-existent, it does not necessarily mean we will have relative extrema!

In summary:

- A critical number is a value of x where $f'(x) = 0$ or $f'(x) \nexists$. The x must be in the domain of f to be a critical number.
- If a function has relative extrema, they must occur at critical numbers.
- Just because a function has critical numbers, it does not necessarily mean the function has relative extrema.

6.1.2 Back to absolute extrema

We started this discussion by taking about absolute extrema. The Extreme Value Theorem tells us that a continuous function on a closed interval must have an absolute maximum and an absolute minimum function value. These absolute extrema can occur at the endpoints of the interval or at some critical number inside the interval–that's why we went into the discussion of critical numbers and relative extrema.

The Extreme Value Theorem did not tell us how to find the absolute extrema. Now, we know how to determine if a function has relative extrema so we can find the absolute extrema if they occur inside the interval. If they occur at the endpoints, all we need is function values at the endpoints.

In the end, the location of the absolute extrema of a continuous function on a closed interval can vary from problem to problem. Sometimes, both absolute extrema will be at the endpoints. Other times they will both occur inside the interval at critical numbers. Still other times they will occur at some combination of endpoints and critical numbers.

So... here's the procedure for finding the absolute extrema of a continuous function on a closed interval:

- Differentiate the function and find the critical numbers.
- Find function values at any critical numbers *in the interval* and function values at each endpoint. If a critical number is not in the interval, just ignore it.
- The largest function value you get is the absolute maximum function value and the smallest function value is the absolute minimum function value.

You don't have to know if the critical numbers inside the interval represent relative maximums or minimums... all you are looking for is the size of the function values.

Example 1

Find the absolute extrema (if any) of $f(x) = 4x - 1$ on $(-\infty, 8]$.

First note that the interval is not closed so we cannot use the Extreme Value Theorem (EVT). The graph of f is a line with a positive slope which extends infinitely in the negative direction but ends at $f(8) = 31$.

Therefore the absolute maximum is 31 and there is no absolute minimum.

Example 2

Find the absolute extrema (if any) of $f(x) = 4x - 1$ on $(5, 8)$.

Because the graph of f is a line and the endpoints are open, there is no absolute maximum or minimum.

Although it may look like this function as absolute extrema, it does not–we can get larger and larger function values by getting closer and closer to $x = 8$. It has no absolute minimum for similar reasons.

Example 3

Find the absolute extrema (if any) of $f(x) = \dfrac{x}{x+1}$ on $[1, 2]$.

This function is continuous on $[1, 2]$ so the EVT holds.

First find critical numbers.

$$f'(x) = \frac{1}{(x+1)^2}$$

$$f'(x) \neq 0 \text{ and } f'(x) = 0 \longrightarrow x = -1$$

-1 is not in the domain of f and is therefore not a critical number.

Since there are no critical numbers in the interval, the absolute extrema must occur at the endpoints.

$$f(1) = \frac{1}{2} \text{ and } f(2) = \frac{2}{3}$$

Thus, the absolute maximum function value is $\frac{2}{3}$ and the absolute minimum function value is $\frac{1}{2}$.

Example 4

Find the absolute extrema (if any) of $f(x) = 4x^3 - 15x^2 + 12x$ on $[0, 3]$.

This is a continuous function on a closed interval so the EVT holds.

$$f'(x) = 12x^2 - 30x + 12$$

$$f' \exists \forall x \text{ and } f'(x) = 0 \longrightarrow x = \frac{1}{2} \text{ or } x = 2$$

Both of these critical numbers are in $[0, 3]$ so we will find function values at both endpoints and at both critical numbers.

$$f(0) = 0 \text{ and } f(3) = 9 \text{ and } f(2) = -4 \text{ and } f\left(\frac{1}{2}\right) = \frac{11}{4}$$

Therefore, f has an absolute maximum function value of 9 and an absolute minimum function value of -4.

6.2 The Mean Value Theorem

6.2.1 Introduction

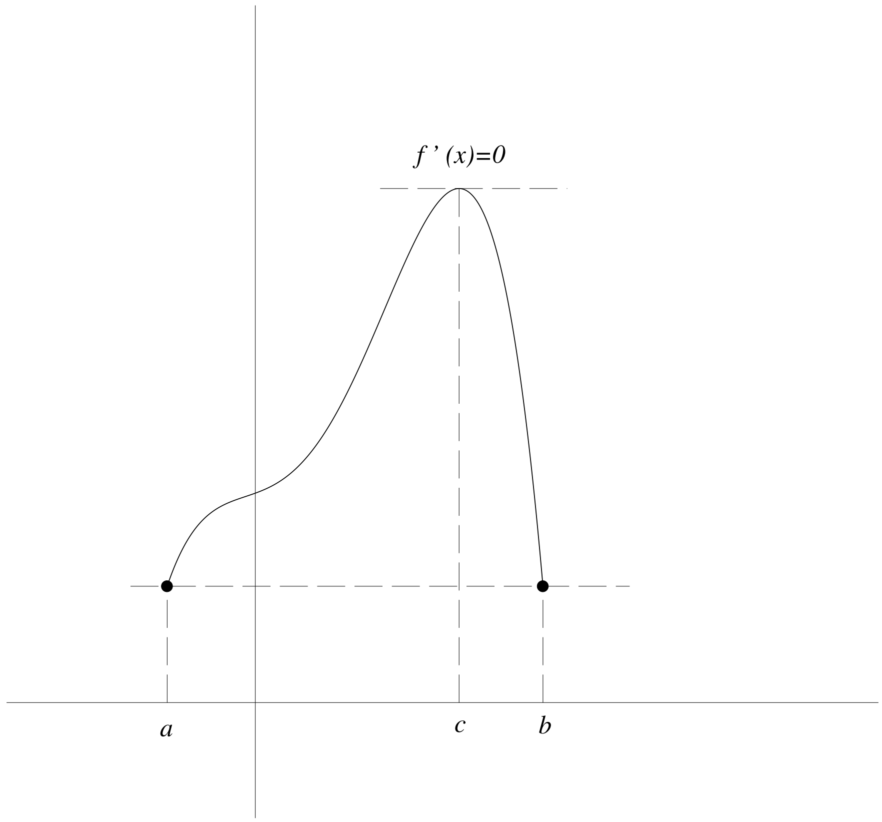
The Mean Value Theorem, like the Intermediate Value Theorem and the Extreme Value Theorem, is another "existence" theorem. It will guarantee us that certain numbers exist. The Mean Value Theorem is actually used quite often used in proving other theorems in calculus. It will tell us something very important about the relationship between the average rate of change in function values on an interval and the instantaneous rate of change in function values at some point in the interval. Before we get to the Mean Value Theorem, we will discuss one if it's corollaries, Rolle's Theorem.

Rolle's Theorem

If f is continuous on $[a, c]$ and differentiable on (a, b) and if $f(a) = f(b)$, then there exists a $c \in (a, b)$ such that $f'(c) = 0$

Why does the theorem state that f must be continuous on $[a, b]$ and not just (a, b)? This is because we need the function values at the endpoint to be equal and so they must exist. The function must be differentiable on (a, b) simply because a derivative cannot exist at and endpoint–we would always be missing either a derivative from the left or right.

The theorem basically tells us that if a function is well-behaved on some interval and the function values at the endpoints are equal, then the function reaches either a maximum or minimum at least once in the interval. The graph below shows one possible example of the theorem.



Example 1

Verify that Rolle's Theorem holds for $f(x) = x^3 + x^2 - 2x + 1$ on $[-2, 0]$, then find the value of c that satisfies the conclusion of the theorem.

Since f is a polynomial function, it is continuous everywhere, thus it is continuous on $[-2, 0]$.

$$f'(x) = 3x^2 + 2x - 2$$

The derivative is also a polynomial therefore f is differentiable everywhere, thus it is differentiable on $(-2, 0)$.

$$f'(x) = 0 \longrightarrow x = -1.215 \text{ or } x = .549$$

Now, $x = .549$ is not in $(-2, 0)$ so $c = -1.215$ only.

Of course, questions about Rolle's Theorem can come in many forms. You will need to recognize the theorem when you read the question. A typical AP question might sound something like this: "If f is a function that is continuous on $[3, 6]$ and differentiable on $(3, 6)$ and $f(3) = f(6)$, which of the following statements must be true?" You would be looking for a choice that states "$f'(c) = 0$ for some c in $(3, 6)$."

Historical note:
Rolle's Theorem, named for the French mathematician Michel Rolle, first appeared in 1691. The interesting irony is that after discovering this theorem, Rolle became a fierce critic of calculus. Rolle, and many other mathematicians of the day were very opposed to the continued work on the calculus because of the "supress the h" step that had to be used when finding a derivative until we got a formal definition of limit. In fact, Rolle declared that calculus was a "...collection of fallacies." It is truly ironic that despite his vehement opposition to calculus, his name and theorem now appear in every calculus text ever written.

6.2.2 The Mean Value Theorem for derivatives

The Mean Value Theorem for Derivatives

If f is continuous on $[a, b]$ and differentiable on (a, b), then,

there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The left side of the conclusion $f'(c)$, is the instantaneous rate of change in f at $x = c$. The right side $\frac{f(b)-f(a)}{b-a}$ is the average rate of change in function values over the interval from $x = a$ to $x = b$. The theorem tells us that for a well-behaved function, on any interval, there is some point in the interval where the average rate of change and the instantaneous rate of change are equal.

This theorem is another of our "existence" theorems and it does have some interesting applications. Consider a trip along a toll road. You enter through a tollbooth were you get a piece of paper with the time stamped on it. You travel to the other end, stop at the tollbooth and hand the person your piece of paper. They put it into their time stamp machine. Well, they know how long the road is and they know the speed limit. Knowing these two things and the time you took to go from one booth to the other, they know if you exceeded the speed limit or not!

Let's say the road is 100 miles long and the speed limit is 50 miles per hour. If you start at noon, you should arrive at the end at 2:00 p.m. (or later). If you finish at 1:55 p.m., your average velocity will be more than 50 miles per hour. Since your average velocity and your instantaneous velocity must be equal at some point, at some point you exceeded 50 miles per hour...and that's why you got a ticket even though you never saw any police!

Example 2

Verify that the Mean Value Theorem holds for $f(x) = 2x^3 + x^2 - x - 1$ on $[0, 2]$ and then find the c that satisfies the conclusion of the theorem.

f is a polynomial so it is continuous on $[0, 2]$.

$$f'(x) = 6x^2 + 2x - 1$$

f' is a polynomial so f is differentiable on $(0, 2)$.

Therefore the theorem holds.

Now, we will solve

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$6c^2 + 2c - 1 = \frac{17 - (-1)}{2 - 0}$$

$$6c^2 + 2c - 10 = 0 \longrightarrow x = -1.468 \text{ or } x = 1.135$$

Since c must be in $(0, 2)$, $c = 1.135$.

Like Rolle's Theorem, the questions are not always straightforward. You will need to be able to recognize the theorem in a question. For example: "If g is continuous on $[2, 7]$ and differentiable on $(2, 7)$, which of the following statements must be true?" The answer you are looking for may look like, "$5g'(x) = g(7) - g(2)$ for some c in $(2, 7)$" Note that if you divide both sides by 5, you get the conclusion of the Mean Value Theorem.

You will see questions like those discussed above on the AP test! Know your theorems! I find it helpful to know the diagram that goes with the theorem... if I know the diagram, I know the theorem.

Historical note:
The Mean Value Theorem was first formulated by Joseph-Louis Lagrange (1736-1813). He was born in Italy to French and Italian parents but spent a large portion of his life in France. His mentor was actually Leonhard Euler! Lagrange, one of history's greatest mathematicians, was both an astronomer and a mathematician who made significant contributions to calculus, the theory of functions, number theory and celestial mechanics to name a few. He succeeded Euler as professor of mathematics at the Berlin Academy at the tender age of 19. After spending some time at the Berlin Academy, he accepted an appointment from France's King Louis XVI and actually lived in the Louvre–which make sense because at that time the Louvre was a palace! History tells us that, unlike some mathematicians of his day, Lagrange was a kind, gentle man who dedicated his entire life to science and mathematics.

6.3 Derivatives and the Analysis of a Function

6.3.1 Introduction

This section is composed of several topics:

- The first derivative test for relative extrema
- Concavity and inflection points
- The second derivative test for relative extrema
- Putting it all together–the analysis of a function
- Finding the graph of f from the graph of f' and visa versa

This material represents several sections in a traditional calculus textbook. I have lumped them all together because the concepts are so closely related. What we are about to do is use the first and second derivative to analyze a function. By "analyze" I mean we will be able to determine on what intervals a function is increasing or decreasing, where it has relative extrema, what type of extrema they are, on what intervals the curve is concave up or down and where it has inflection points. Now, I realize that you don't have a clue yet about concavity or inflection points but don't worry... you'll know all about it very soon! One more item... we will also learn about a third type of asymptote–the oblique asymptote.

6.3.2 The first derivative test for relative extrema

We already know that if a function has relative extrema, they must occur at critical numbers. The first derivative test will tell us if there is an extremum and will also tell us if it is a relative maximum or a relative minimum. Let's review a couple of definitions before we begin.

A function f is said to be **increasing** on the interval I if and only if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ where both x_1 and x_2 are both in I.

A function f is said to be **decreasing** on the interval I if and only if $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ where both x_1 and x_2 are both in I.

A function f has a **relative (local) minimum** at $x = c$ if there exists an open interval I containing c such that $f(c) \leq f(x)\ \forall\ x \in I$.

A function f has a **relative (local) maximum** at $x = c$ if there exists an open interval I containing c such that $f(c) \geq f(x)\ \forall\ x \in I$.

The First Derivative Test for Relative Extrema

If c is a critical number of f:

If $f'(c) < 0\ \forall\ x$ in some interval having c as its right endpoint and $f'(c) > 0\ \forall\ x$ in some open interval having c as its left endpoint, then f has a relative (local) minimum of $f'(c)$ at $x = c$.

If $f'(c) > 0\ \forall\ x$ in some interval having c as its right endpoint and $f'(c) < 0\ \forall\ x$ in some open interval having c as its left endpoint, then f has a relative (local) maximum of $f'(c)$ at $x = c$.

OK, the definition is a little hard to read. Here's what it means... if the derivative is positive (f is increasing) before your critical number and negative (f is decreasing) after it, there is a relative maximum at the critical number. If the derivative is negative (f is decreasing) before your critical number and positive (f is increasing) after it, there is a relative minimum at the critical number.

Example 1

Find the relative extrema (if any) of $f(x) = x^3 - 9x^2 + 15x - 5$.

$$f'(x) = 3x^2 - 18x + 15$$

$$f' \exists \forall x$$

$$f'(x) = 0 \longrightarrow x = 1 \text{ or } x = 5.$$

At this point you can make a sign chart to find out where the derivative is positive or negative.

Better still is to just look at the derivative itself. It is an upward opening parabola that crosses the axis at $x = 1$ and $x = 5$. Just from the sketch you can see where f' is above (positive) and below (negative). This technique can be used any time you can easily sketch the derivative... like with factorable cubics and quadratics. *It is easier and faster and you should use it whenever you can!*

Since $f'(x) > 0$ on $(-\infty, 1)$ and $f'(x) < 0$ on $(1, 5)$, f has a relative maximum of 2 at $x = 1$.

Since $f'(x) < 0$ on $(1, 5)$ and $f'(x) > 0$ on $(5, \infty)$, f has a relative minimum of -30 at $x = 5$.

Example 2

Find the relative extrema (if any) of $f(x) = \dfrac{x-2}{x+2}$.

$$f'(x) = \frac{4}{(x+2)^2}$$

$f' \nexists$ at $x = 2$ but $x = 2$ is not in the domain of f so it is not a critical number.

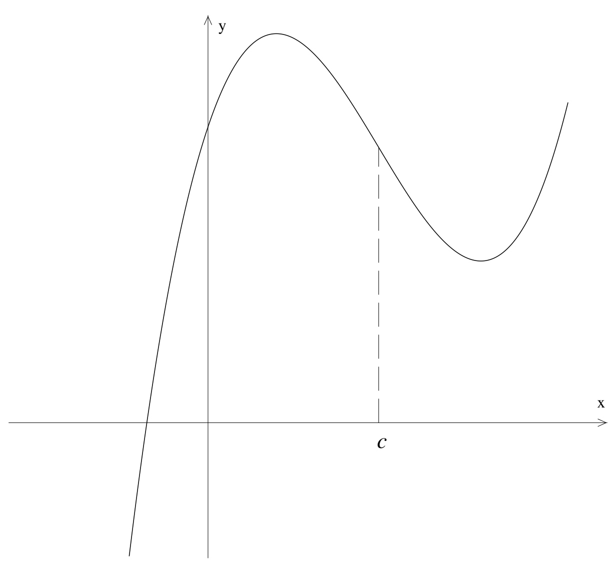
$$f'(x) \neq 0$$

$4 > 0$ and $(x+2)^2 \geq 0 \; \forall \; x$ therefore $f'(x) > 0 \; \forall \; x$ in f.

f is therefore always increasing and has no relative extrema.

6.3.3 Concavity and points of inflection

Consider the graph of a function f below.



Notice that on $(-\infty, c)$ any tangent line drawn will lie above the curve. The curve is called "concave down" on $(-\infty, c)$. Now note that on (c, ∞) any tangent drawn will lie below the curve. The curve is called "concave up" on (c, ∞).

Let's examine a little further. On $(-\infty, c)$ the value of f' decreases as x gets closer and closer to c. Since the value of f' is getting smaller, $f'' < 0$ because just as f' tells us the rate of change in f, f'' tells us the rate of change in f'. Now as the value of x moves away from c to the right, the value of f' gets larger (until we get to the minimum function value, $f' < 0$, then after the minimum $f' > 0$.) Since the value of f' is getting larger, $f'' > 0$.

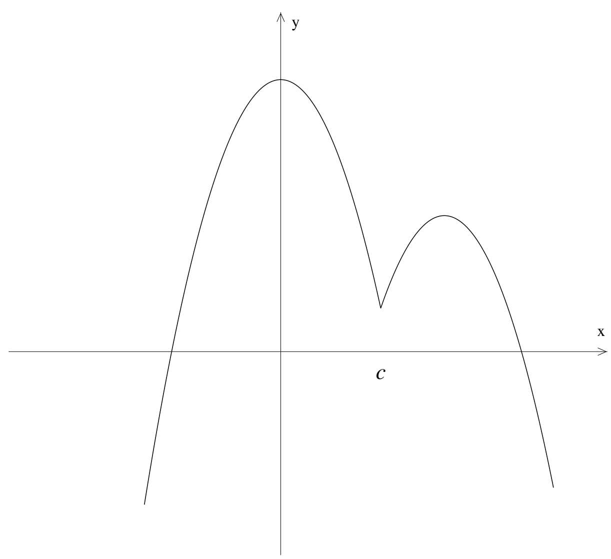
If $f''(x) < 0$ on (a, b), f is **concave down** on (a, b).

If $f''(x) > 0$ on (a, b), f is **concave up** on (a, b).

Back to our sketch. Note that $f''(x) < 0$ on $(-\infty, c)$ and $f''(x) > 0$ on (c, ∞). The Intermediate Value Theorem tells us then, that $f''(x) = 0$ at some point. That point occurs at $x = c$. A point on a curve where the concavity changes is called an **inflection point**.

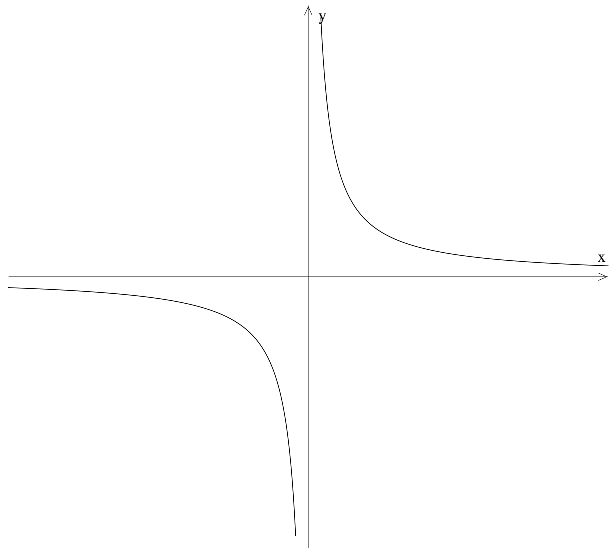
When we look for inflection points, we look for values of x where $f''(x) = 0$ or f'' does not exist…just like critical numbers only we use the second derivative. When we use the second derivative however, we *do not* call them critical numbers–that terms is only for the first derivative. Values of x where the second derivative is zero or nonexistent are simply called "possible inflection points".

Just like critical numbers, just because the second derivative is zero or nonexistent, does not mean we must have inflection points. Consider the sketch below.



We clearly have a place where $f'' \nexists$, $x = c$. But note that $f''(x) < 0$ both before and after c…thus there is no inflection point.

Now take a look the the next sketch.



In this case, $f''(x) \nexists$ at $x = 0$ but $f''(x) < 0$ before $x = 0$ and $f''(x) > 0$ after c. Here we have a change in concavity but no inflection point!

In the end, when finding inflection points you must find places where f'' is zero or non-existent and then you must make sure that f'' changes sign *and* you must make sure there is a function value at that point.

Example 3

Determine where the graph of $f(x) = x^3 - 3x + 1$ is concave up and concave down. Find any inflection points.

$$f'(x) = 3x^2 \longrightarrow f''(x) = 6x$$

$$f'' \exists \ \forall\, x \text{ and } f''(x) = 0 \text{ when } x = 0$$

(Again, we don't really need a sign chart here…the second derivative is just a line with a positive slope and a y-intercept of 0.

Since $f''(x) < 0$ on $(-\infty, 0)$, f is concave down on $(-\infty, 0)$.

Since $f''(x) > 0$ on $(0, \infty)$, f is concave up on $(0, \infty)$.

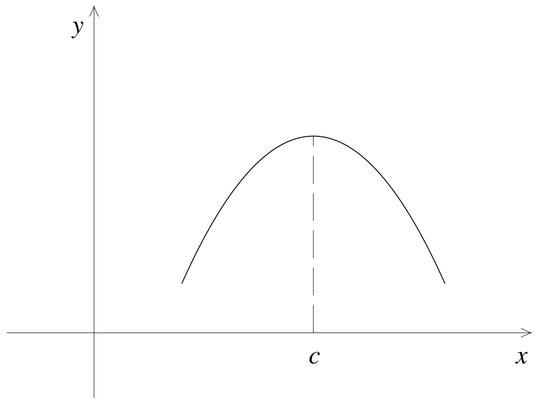
Since $f''(x) < 0$ on $(-\infty, 0)$ and $f''(x) > 0$ on $(0, \infty)$ and $f(0) = 1$, f has an inflection point at $(0, 1)$.

6.3.4 The second derivative test for relative extrema

First off…the second derivative test is not a test for concavity. It is a test for relative (local) extrema–and it's very fast and clean for many functions.

Quick note…at times we want to refer the interval just before and just after some x value. We call this the "delta-deleted neighborhood of x". The "delta" is the same notion of "delta x" we used when we learned about limits.

Let's look at a local maximum.



In this situation we have a critical number at $x = c$. We can also clearly see that the curve is concave down in the delta-deleted neighborhood of c. It's pretty obvious that this function has a local maximum at $x = c$.

Similarly, if $f''(x) > 0$ in the delta-deleted neighborhood of c, then the curve is concave up and the function has a local minimum at $x = c$.

The test is very easy to perform and is the preferred method to find relative extrema when it can be used.

The Second Derivative Test for Relative Extrema

If f has a critical number at $x = c$ and $f''(c) > 0$, then f has a relative (local) minimum at $x = c$.

If f has a critical number at $x = c$ and $f''(c) < 0$, then f has a relative (local) maximum at $x = c$.

Note that the test will not work if $f''(c)$ is zero or non-existent. We need the second derivative to be either a positive or negative number to use the second derivative test. In this case you must use the first derivative test.

Example 4

Use the second derivative test to find the relative extrema (if any) of $f(x) = x^3 - 9x^2 + 15x - 5$.

$$f'(x) = 3x^2 - 18x + 15$$

$$f' \exists \ \forall \, x \text{ and } f'(x) = 0 \ \longrightarrow \ x = 5 \text{ or } x = 1$$

$$f''(x) = 6x - 8$$

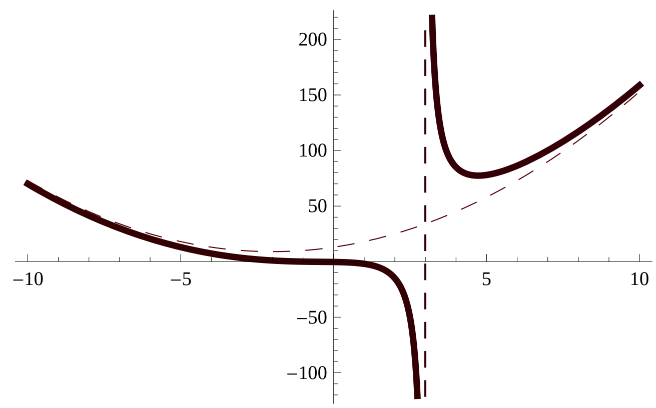
Since $f''(5) = 12 > 0$ f is concave up at $x = 5$ therefore f has a relative minimum of $f(5) = -30$ at $x = 5$.

Since $f''(1) = -12 < 0$ f is concave down at $x = 1$ therefore f has a relative maximum of $f(1) = 2$ at $x = 1$.

6.3.5 Analysis of a function (curve sketching)

Years ago, calculus was the only way to get a reasonably accurate sketch of a function. Increasing, decreasing, relative extrema, concavity and inflection point could only be found by someone versed in calculus. That is where the term "curve sketching" came from. With the advent of computers and graphing calculators, anyone who can punch keys can obtain a very accurate graph of a function. The emphasis in more recent years is more on using calculus to *explain why* a curve looks the way it does.

Before we move on, there is one more small item you need to know about. . . oblique (or slant) asymptotes. All of the asymptotes we've run into thus far have been horizontal or vertical lines. Asymptotes, in fact, can be linear, quadratic, cubic or any other degree curve. The curve below has both a vertical and a quadratic asymptote.



We get oblique asymptotes when the degree of the numerator is exactly one larger than the degree of the denominator. To find the equation of the oblique asymptote (or any higher degree asymptote), divide the numerator by the denominator and omit the remainder. Consider the following function.

$$f(x) = \frac{2x^3 + 6x^2 + 2}{x^2 + 1}$$

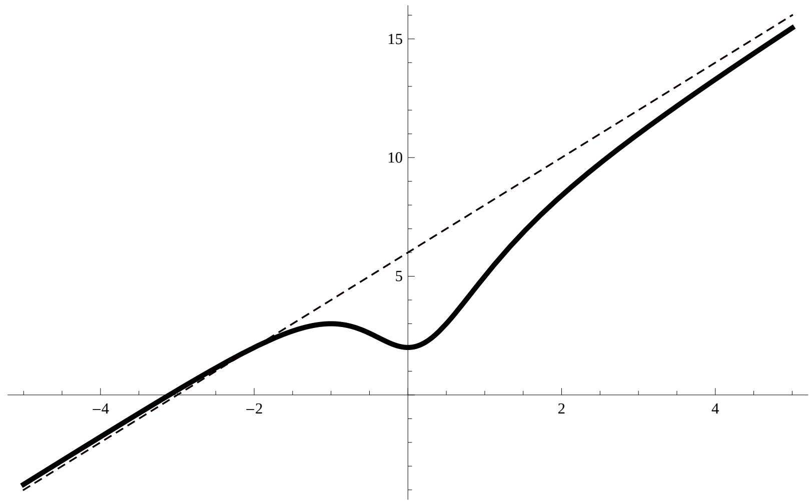
If you did polynomial long division you would get

$$2x + 6 - \frac{2x + 4}{x^2 + 1}$$

Thus the equation of the oblique asymptote is $y = 2x + 6$.

The reason we can omit the remainder is that as $x \to 0$ the limit of the remainder is zero and the function approaches $2x + 6$.

A graph of f is shown below.



Let's get back to the analysis of a curve. This entails finding the following items:

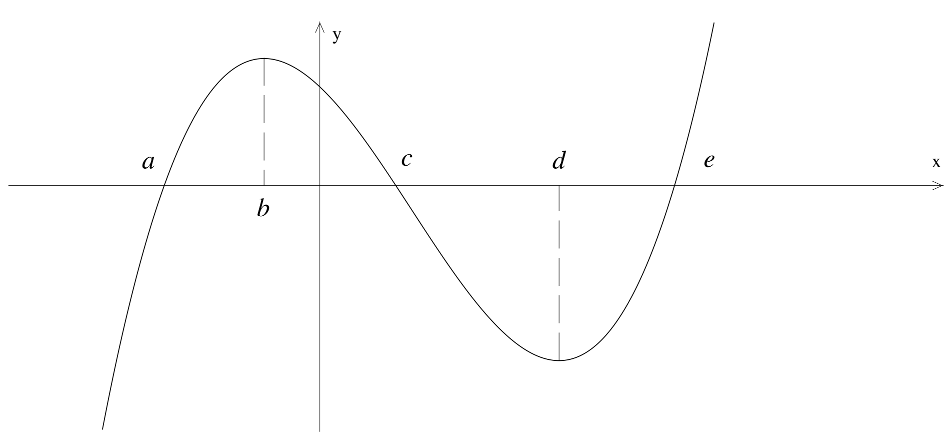
- Finding the domain
- Finding zeros
- Finding any vertical, horizontal and oblique asymptotes
- Determining on what intervals the function is increasing or decreasing
- Determining on what intervals the function is concave up or concave down
- Finding any inflection points

Fortunately for us, we rarely have to do all of these in the same problem but you should be able do any of the above items with any function you are given. "Curve sketching" and "analysis of a curve" sections in most calculus text just involve the student being able to determine the items we listed.

6.3.6 Graphs f and f'

One of the classic problems in a first year calculus course is sketching the graph of a function when given the graph of its derivative and visa versa. Consider the following diagram.

Graph of f'



So, what can we tell about f from the graph of f'?

- **Increasing and decreasing**
  - f' is below the axis on $(-\infty, a)$ and (c, e) $\longrightarrow$ f is decreasing on $(-\infty, a)$ and (c, e)
  - f' is above the axis on (a, c) and (e, ∞) $\longrightarrow$ f is increasing on (a, c) and (e, ∞)
- **Relative(local) extrema**
  - At $x = a$ and $x = e$, f' changes from negative to positive below to above the axis) $\longrightarrow$ f has relative maximums at $x = a$ and $x = e$
  - At $x = c$, f' changes from positive to negative above to below the axis) $\longrightarrow$ f has a relative minimum at $x = c$
- **Concavity**
  - f' is increasing on $(-\infty, b)$ and (d, ∞). Since f' is increasing, $f'' > 0$ $\longrightarrow$ f is concave up on $(-\infty, b)$ and (d, ∞).
  - f' is decreasing on (b, d). Since f' is decreasing, $f'' < 0$ $\longrightarrow$ f is concave down on (b, d).
- **Inflection points**
  - Note the relative maximum on f' at $x = b$. This means that f'' changed from positive to negative at $x = b$ $\longrightarrow$ f has an inflection point at $x = b$.
  - There is a relative minimum at $x = c$. This means that f'' changed from negative to positive at $x = d$ $\longrightarrow$ f has an inflection point at $x = d$.
  - Where f' has relative extrema, f has inflection points.

So, if you are given the graph of f', you should be able to sketch a possible graph of f... and visa versa!

In general:

- If f' is above the axis, f is increasing.
- If f' is below the axis, f is decreasing.
- If f' goes from below to above the axis, f has a relative minimum.
- If f' goes from above to below the axis, f has a relative maximum
- If f' is increasing, f is concave up.
- If f' is decreasing, f is concave down.
- If f' has relative extrema, f has inflection points.

6.4 Applied Maximum and Minimum Problems (Optimization Problems)

6.4.1 Introduction

Optimization problems are an application of the Extreme Value Theorem. Remember, this theorem tells us that if a function is continuous on a closed interval, it must have both an absolute maximum and an absolute minimum function value. The absolute extrema must occur at endpoints or at critical numbers. Here's the good news... most of the time we do not need the endpoints. We will only need the endpoints if the function value at our critical number makes no sense.

The basic process is simple:

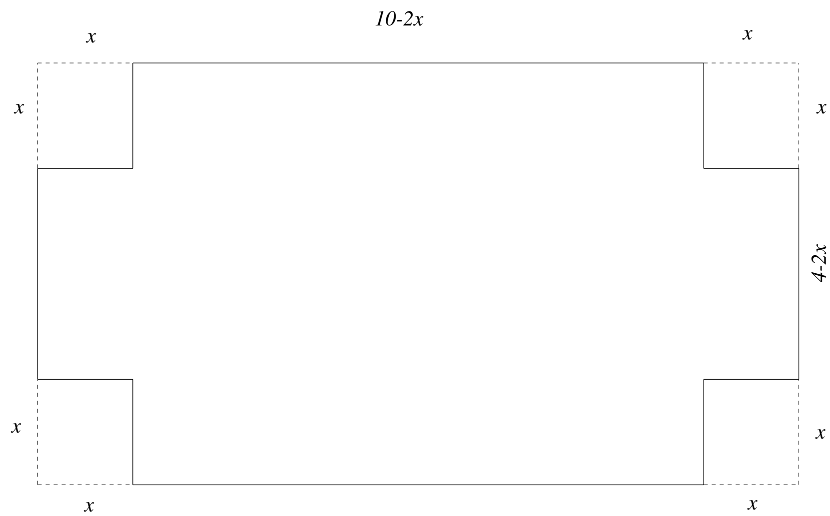
- Determine what quantity is to be optimized.
- Draw a diagram if at all possible.
- Find a function, in a single variable, which describes this quantity.
- Take a derivative and find critical numbers.
- Find function values at your critical numbers (usually there will be only one!
- Answer the question!

Example 1

A box with an open top is to be constructed from a sheet of cardboard 4 feet wide and 10 feet long by cutting squares from each of the four corners and folding up the sides. Find the largest volume that such a box can have.

We need to maximize the volume, so we need a function in a single variable which yields the volume of any such box.

First we begin with a diagram.



From our diagram we can see that the volume of the resulting box is

$$V(x) = x(4-2x)(10-2x) \longrightarrow V(x) = 4x^3 - 28x^2 + 40x$$

$$V'(x) = 12x^2 - 56x + 40$$

$$V' \exists \ \forall\, x \text{ and } V'(x) = 0 \longrightarrow x = .880 \text{ or } x = 3.786$$

Now, we obviously cannot cut 3.786 feet from each corner…the box is only 4 feet wide!

Therefore we will cut .880 feet from each corner.

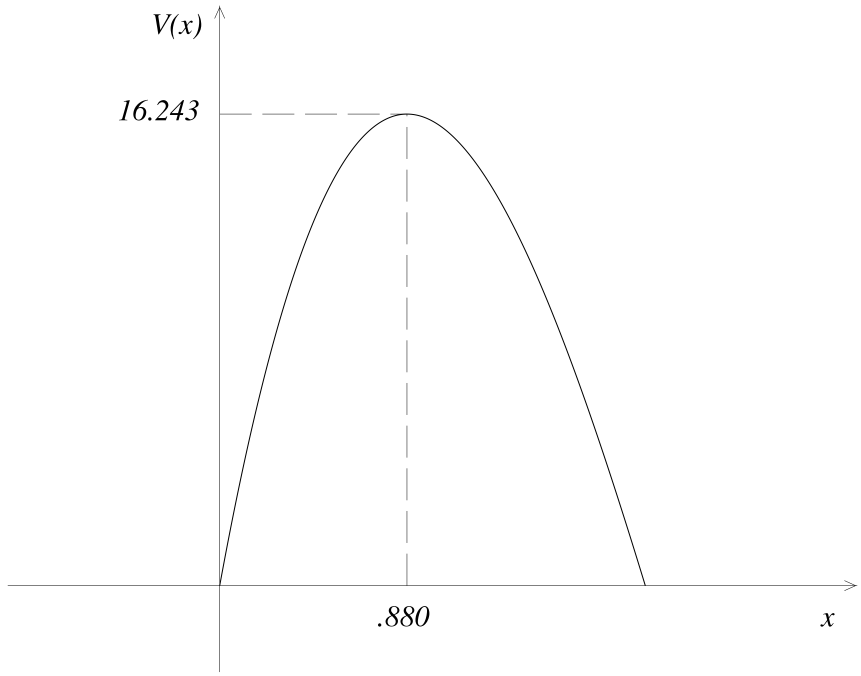
$$V(.880) = 16.243$$

Therefore the maximum volume possible is 16.243 cubic feet.

Sometimes our answer is our critical number. In Example 1, the question could have been, "Find the size of the corners to be cut to maximize the volume." In, this case our answer would have been, "The maximum volume is obtained by cutting .880 feet from each corner."

IMPORTANT: As with all word-type problems, we **never put units on our mathematics!** If there are units in the problem, we *must* put units in our answer. Therefore always *answer the question, and always put units in the answer*, not on your mathematics!

Before we move on to the next example, let's take a look at what happened in Example 1 graphically. The largest possible cut we could have made was 2 feet. Silly, because we end of up four separate pieces of cardboard…not much of a box. The smallest cut we can theoretically make is 0. Again, not much of a box! So, technically we were finding the absolute maximum of V on $[0, 2]$. Let's take a look at V on this interval.



All we really did was find a maximum function value using the derivative... old stuff really. So how do you know if you've found a maximum or minimum? It doesn't matter... the problems are set such that you will automatically find the correct one!

Example 2

Find two numbers whose difference is 100 and whose product is a minimum.

We need to minimize the product so we need a function in a single variable which describe the product.

A diagram is not very useful on this problem.

If two numbers differ by 100, we can call one of them x and the other $x + 100$. ($x - 100$ would also work.)

Their product is written: $P(x) = x(x + 100)$

$$P(x) = x^2 + 100x \longrightarrow P'(x) = 2x + 100$$

$$P' \exists \ \forall x \text{ and } P'(x) = 0 \longrightarrow x = -50$$

In this case we need the numbers not the product so the numbers are -50 and 50.

Example 3

Find the point on the parabola $x^2 + y = 0$ that is closest to the point $(0, -3)$.

We need to minimize the distance so we need the distance formula.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Here's something to remember when dealing with this type of distance problem. Minimizing the square root of a number is the same as minimizing the number itself. Smaller numbers have smaller square roots. The same goes for maximizing. What this means is that can use the distance formula without the radical and get the same results.

Now, we can do this problem in terms of x or y but because we are given $x^2 + y = 0$ we should do the problem in y so that we do not have to take a square root to solve it.

Solving for x yields $x = -y^2$ so any point on the parabola can be written $(-y^2, y)$.

We will now use the distance formula on the two points $(-y^2, y)$ and $(0, -3)$.

Remember, we do not need the radical part of the distance formula!

$$D(y) = (-y^2 - 0)^2 + (y + 3)^2$$

$$D(y) = y^4 + y^2 + 6y + 9 \longrightarrow D'(y) = 4y^3 + 2y + 6$$

$$D' \exists \ \forall x \text{ and } D'(y) = 0 \longrightarrow y = -1$$

Substituting $y = -1.$ into the equation of the parabola gives us $x = -1$

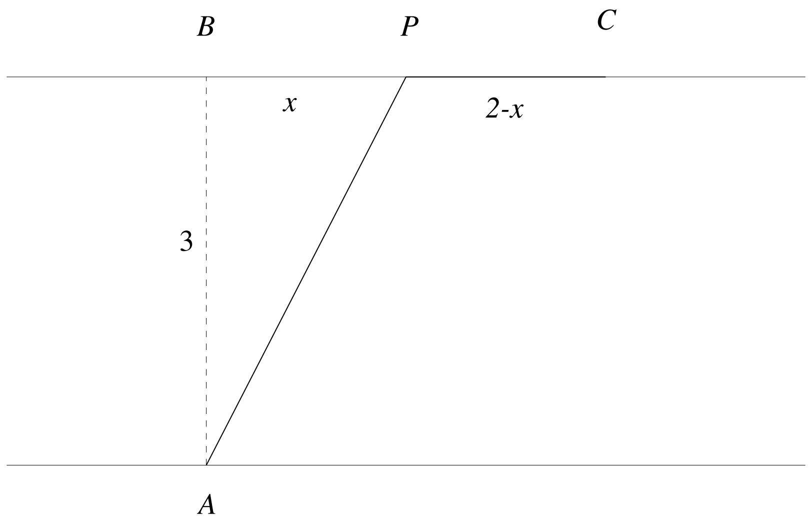
Therefore, the point on the parabola that is closest to $(0, -3)$ is $(-1, -1)$.

Example 4

Points A and B lie on opposite sides of a river 3 km wide. Point C is on the same side as B but is 2 km downstream. A cable company wants to lay a cable from A to C. The cost of laying the cable under water is \$12,000 per kilometer and the cost of laying the cable over land is \$10,000 per kilometer. How should the cable company lay the cable so as to minimize the cost?

First, consider the possibilities. They can lay the cable directly from A to C (entirely underwater). This may be the shortest path but may not be the least expensive. They could lay the cable from A to B, then overland to C. This minimizes the underwater length but may not be the least expensive–it is the longest path. Finally they could lay the cable from A to some point P between A and C (under water) and then overland from P to C. We must consider this third possibility first.

If we call the distance from B to P x, then the distance from P to C will be $2 - x$. By the Pythagorean Theorem, the distance from A to P will be $\sqrt{9 + x^2}$. Here is the diagram.



Now that we have our distances labeled, we can get a cost function in terms of x.

Total Cost= (cost under water)(length under water)+(cost over land)(length over land)

$$C(x) = 12500\sqrt{9+x^2} + 10000(2-x)$$

$$C'(x) = \frac{12500x - 10000\sqrt{9+x^2}}{\sqrt{9+x^2}}$$

$$C' \exists \ \forall\, x \text{ and } C'(x) = 0 \ \longrightarrow\ x = 4$$

Now, $x = 4$ makes no sense. It is only 2 km from A to C! If $x = 4$ we would be going 2 km downstream of C and back to C. Clearly this critical number makes no sense so we will now test the endpoints of the interval containing x. The shortest we can make x is zero. The biggest we can make x is 2. The interval then is $[0, 2]$.

$$C(0) = 57000 \text{ and } C(2) = 45,069.39$$

Therefore the minimum cost is achieved by laying the cable directly from A to C–entirely underwater.

Chapter 7

Antidifferentiation

7.1 Introduction

Antidifferentiation is the process of finding a function whose derivative is given. Thus far in the course you have been given a function and you found the derivative. We now turn this process around… you'll be given the derivative and be asked to find the "original" function. Let's get this out there right away… antidifferentiation can be much more, should we say, subtle than differentiation. Unlike differentiation, we do not have a simple set of rules we follow. There is no quotient rule, product rule or chain rule for antiderivatives. Instead, we have a power rule, some known antiderivatives and then we have techniques we use. The good thing is that in this course the only technique we learn is something called "substitution", or "change of variable". There are other techniques you will learn in later calculus courses. Techniques like integration by parts, trigonometric substitution and integration by partial fractions.

Another important difference between differentiation and antidifferentiation is that with differentiation, you start with a function, differentiate it, and get a single function. When we antidifferentiate a function, we get an entire family of functions! We'll learn about that very soon.

Eventually you will hear the terms "antidifferentiation" and "integration" used somewhat interchangeably. They are, in fact, different but you will quite often even hear me say, "OK, let's integrate...". Don't worry about it right now, we'll get there soon enough.

7.1.1 Antiderivatives

Let's start with a definition.

A function F is called the antiderivative of f on an interval I if

$$F'(x) = f(x) \; \forall \; x \; \in I$$

The notation we use comes directly from Leibniz (whose calculus was much easier to understand than Newtons's). The definition can also be written as shown below.

$$\int f(x)\ dx = F(x) + C \text{ if } F'(x) = f(x)\ \forall\ x$$

The symbol $\int$ is actually an elongated "S" which stands for "sum". Why we use this notation will become clearer once we have a better understanding of integration. For now, think of the $\int$ and the dx as bookends which contain the function you are working with. The problem

$$\int x^2\ dx$$

is asking you to find all the functions whose derivative is x^2. In general,

$$\int f(x)\ dx$$

means "find the functions whose derivative is $f(x)$". The "$+C$" on the end will be explained in a moment.

We will start with finding antiderivatives of functions which we obtained using only the power rule for derivatives. We will need slightly more sophisticated methods to handle functions which were obtained using the chain rule. As was mentioned above, the process of antidifferentiation is made up more of techniques than theorems. We do have one theorem to get us started however, the power rule for antiderivatives.

Before we state the power rule for antiderivatives, let's look at a problem.

Let's suppose you were told that some function was differentiated and the result was $3x^2$. You need to find the function which was differentiated. Using our new notation, this problem would be stated

$$\int 3x^2\ dx.$$

Well, when we differentiated using the power rule for derivatives first you multiply the function by the exponent, then subtract one from the exponent to get your new exponent. When we antidifferentiate, we do the opposite operations and in the reverse order! So, to antidifferentiate $3x^2$, first we add one to the exponent to get our new exponent. Then we divide by the new exponent. The result would be

$$\frac{3x^3}{3} = x^3.$$

There's just one problem, although the derivative of x^3 is $3x^2$, which is exactly what we wanted, the derivative of $x^3 + 8$ is also $3x^2$. The derivative of $x^3 - 5$ is also $3x^2$. In fact, there are an infinite number of functions whose derivative is $3x^2$, and the all differ by a constant. To show all the antiderivatives, we add a "$+C$" to the end of our antiderivative. When we differentiate a function, we get a one function. When we antidifferentiate a function, we get a family of functions! Using the correct notation, our problem now looks like this:

$$\int 3x^2\ dx = x^3 + C$$

The process we just went through was an application of the power rule for antiderivatives. To antidifferentiate a single variable raised to a power, we add one to the exponent, then divide by the new exponent.

Note the the rule only works on a single variable raised to a power. If we have a function raised to a power, we will have to use a different technique before using the power rule.

Does the power rule for antiderivatives work for any exponent. No. Try using the power rule on x^{-1}. Adding one to the exponent and getting a zero is fine, but then we must divide by zero... that's a problem! Well, let's give this some thought. All we want is

$$\int \frac{1}{x}\ dx.$$

We are looking for a function whose derivative is $1/x$. Hopefully we all remember... it's just $\ln x$!

$$\int \frac{1}{x}\ dx = \ln|x| + C$$

We need the absolute value because we need to make sure we never attempt to take a logarithm of a negative quantity. So, the power rule is actually made up of two parts, one part for any exponent that is not -1 and another part if the exponent is -1.

The Power Rule for Antiderivatives

$$\int u^n\ du = \begin{cases} \dfrac{u^{n+1}}{n+1} & \text{for } n \neq -1 \\ \ln|u| & \text{for } n = -1 \end{cases}$$

We use "u" instead of x or $f(x)$ because the "u" can actually represent a function–but when we use the rule, the rule must be applied to a single variable. I know it's a little confusing but once we learn a little more about something called "substitution", it will be clearer. For now, suffice it to say that using "u" makes the notation more flexible.

Example 1

Find $\displaystyle\int x^5\ dx$.

$$\begin{aligned}\int x^5\ dx &= \frac{x^6}{6} + C \\ &= \frac{1}{6}x^6 + C\end{aligned}$$

Example 2

Find $\displaystyle\int (x^2 + 3x - 5\ dx$.

$$\int (x^2 + 3x - 5)\ dx = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 5x + C$$

Note: The antiderivative of a sum or difference is the sum or difference of the antiderivatives. However, like differentiation, the antiderivative of a product or quotient is *not* the product or quotient of the antiderivatives!

Example 3

Find $\int \left(\sqrt[3]{x^2} - \sqrt[3]{x}\right)\, dx$.

First rewrite the problem with rational exponents, then use the power rule.

$$\begin{aligned}\int \left(x^{2/3} - x^{1/3}\right)\, dx &= \frac{3}{5}x^{5/3} - \frac{3}{4}x^{4/3} + C \\ &= \frac{3}{5}\sqrt[3]{x^5} - \frac{3}{4}\sqrt[3]{x^4} + C\end{aligned}$$

Example 4

Find $\int \frac{x^2 + 2x}{\sqrt{x}}\, dx$.

$$\begin{aligned}\int \frac{x^2 + 2x}{\sqrt{x}}\, dx &= \int x^{-1/2}(x^2 + 2x)\, dx \\ &= \int \left(x^{3/2} + 2x^{1/2}\right)\, dx \\ &= \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + C \\ &= \frac{2\sqrt{x^5}}{5} + \frac{4\sqrt{x^3}}{3} + C\end{aligned}$$

Note that this answer is in a slightly different form than Example 3…it really does not matter–you just need to recognize your answer on an AP test.
Also, we do put answers in radical form but there is no need to simplify radicals.

Example 5

Given $f''(x) = 60x^4 - 45x^2$, find $f(x)$.

In this case we just need to antidifferentiate twice so we will have two constants. We do not need to use or usual antidifferentiation notation…just antidifferentiate.

$$f'(x) = 12x^5 - 15x^3 + C$$

$$f(x) = 2x^6 - \frac{15}{4}x^4 + Cx + D$$

Now, any differentiation theorem you know can be used in reverse to find antiderivatives. For instance, you know that the derivative of $\sin x$ is $\cos x$ so the antiderivative of $\cos x$ is $\sin x$. Be careful... right now we are restricting ourselves to functions which did not require the chain rule to differentiate. We can say

$$\int \cos x \; dx = \sin x + C$$

but

$$\int \cos(3x+5) \; dx \neq \sin(3x+5) + C$$

We faced similar situations when we learned to differentiate. To deal with composite functions, we needed the chain rule. To antidifferentiate composite functions we will need a method to "undo" the chain rule. We'll get to that soon.

For now, remember that antidifferentiation theorems operate on a *single* variable only... not on functions.

Listed below are some known antiderivatives.

$$\int e^u \; du = e^u + C$$

$$\int \frac{1}{u} \; du = \ln|u| + C \text{ (Part of our chain rule.)}$$

$$\int \sin u \; du = -\cos u + C$$

$$\int \cos u \; du = \sin u + C$$

$$\int \sec^u \; du = \tan u + C$$

$$\int \sec u \tan u \; du = \sec u + C$$

$$\int \csc u \cot u \; du = -\csc u + C$$

$$\int \csc^2 u \; du = -\cot u + C$$

Any of the above theorems can be proven by simply taking the derivative of the right side.

7.1.2 General and particular solutions

All the antiderivatives we have found so far are called "indefinite" integrals. They are also called "general" solutions–that's why the "$+C$" is there–general solutions represent a family of functions. We can also find something called a "particular" solution if we are given some initial conditions. When we are looking for a particular solution, we need to find the value of C which specifies which member of the family actually passes through the given point (the initial condition).

Example 6

Given $f'(x) = 12x^2 - 24x + 1$ and $f(1) = 2$, find $f(x)$.

$$f(x) = 4x^3 - 12x^2 + x + C$$

$$\text{Since } f(1) = 2 \text{ we get } -2 = 4(1)^3 - 12(1) + 1 + C \longrightarrow C = 5$$

$$\text{Therefore, } f(x) = 4x^3 - 12x^2 + x + 5$$

Example 7

Given $f''(x) = 20x^3 - 10$ and $f'(1) = -5$ and $f(1) = 1$, find $f(x)$.

$$f'(x) = 5x^4 - 10x + C$$

$$\text{Since } f'(1) = -5 \longrightarrow C = 0$$

$$f(x) = x^5 - 5x^2 + D$$

$$\text{Since } f(1) = 1 \longrightarrow D = 5$$

$$\text{Therefore } f(x) = x^5 - 5x^2 + 5$$

Example 8–Very important example!

At any point (x, y) on a curve the slope of a tangent line is given by $4x - 5$. If the curve contains the point $(3, 7)$, find the equation of the curve.

This is our tangent to a curve problem turned on its head. The $4x - 5$ is the derivative!

$$f'(x) = 4x - 5 \longrightarrow f(x) = 2x^2 - 5x + C$$

$$\text{Since } y = 7 \text{ when } x = 3 \longrightarrow C = 4$$

$$\text{Therefore } f(x) = 2x^2 - 5x + 4$$

7.1.3 Rectilinear motion–again!

We've already done quite a few rectilinear motion problems. We now return to them knowing how to antidifferentiate. In all previous rectilinear motion problems, we were always given a position function. We differentiated the position function to get a velocity function and then answered the questions. Now, all we are given is a set of initial conditions. We will know the initial position, the initial velocity and the acceleration (which is always a constant function.) We will then begin with the acceleration function, antidifferentiate it and use the initial conditions to find the constant–we will then have a velocity function.

We will then antidifferentiate again and use the initial conditions to find the constant–we will then have a position function. The questions will remain the same. . . find the max height, find velocity at impact, etc.

The acceleration function is always your starting point. For some problems you will be given an acceleration function. If the object is subject to gravity, you will have to remember the acceleration constants. If the units are metric, the acceleration is -9.8 meters per second per second. If the units are English, the acceleration is -32 feet per second per second. If the object is dropped, thrown, etc., the acceleration is always a constant function, so you will always start with

$$a(t) = -32$$

or

$$a(t) = -9.8.$$

Example 9

A ball is thrown vertically upward with an initial velocity of 48 feet per second from the edge of a cliff 432 feet high. Find the maximum height the ball attains and the velocity of the ball when it hits the ground at the base of the cliff.

Always start this type of problem by listing the initial conditions.

When $t = 0 \longrightarrow s = 432$, $v = 48$ and $a(t) = -32$

$$a(t) = -32 \longrightarrow v(t) = -32t + C$$

Since, $v = 48$ when $t = 0 \longrightarrow C = 48$

Thus $v(t) = -32t + 48$.

Antidifferentiating again yields $s(t) = -16t^2 + 48t + D$.

Since $s = 432$ when $t = 0 \longrightarrow D = 432$

Thus the position function is $s(t) = -16t^2 + 48t + 432$.

As we know, to find the maximum height, we set the velocity equal to zero and solve for t. The the value of the position function at that t.

$$v(t) = 0 \longrightarrow t = \frac{3}{2}$$

$s\left(\frac{3}{2}\right) = 468 \therefore$ the maximum height is 468 feet.

To find the velocity at impact, set the position equal to zero and use that t in the velocity function.

$s(t) = 0 \longrightarrow t = -3.908$ or $t = 6.908$

We use the positive value. $v(6.908) = -173.066$

Therefore the velocity at impact is 172.066 feet per second downward.

Note how the negative is interpreted in the answer!

Also, make sure you use your calculator correctly to avoid rounding error.

Example 10

A particle moves in a straight line and has acceleration $a(t) = 6t+4$. If its initial velocity is -6 centimeters per second and its initial position is 9, find the position function.

$$a(t) = 6t + 4 \longrightarrow v(t) = 3x^2 + 4t + C$$

$$v = -6 \text{ when } t = 0 \text{ so } C = -6$$

$$\text{Thus } v(t) = 3t^2 + 4t - 6.$$

$$s(t) = t^3 + 2t^2 - 6t + D$$

$$s = 9 \text{ when } t = 0 \longrightarrow D = 9$$

$$\text{Therefore } s(t) = t^3 + 2t^2 - 6t + 9.$$

7.2 Antidifferentiation by Substitution

7.2.1 Introduction

Up to this point we have been only able to antidifferentiate relatively simple functions. Remember, when we see $\int f(x)\ dx$, $f(x)$ is the derivative of some function that we are trying to find. We will now examine a technique that allows us to antidifferentiate more comples expressions that are the result of a more complex application of the Chain Rule. Yes, it's true that we always use the Chain Rule but differentiating x^3 is a lot simpler than differentiating $\sec^9\left(e^{\cos x}\right)$. The technique we are about to learn will allow us to "undo" more complex applications of the Chain Rule.

7.2.2 The technique

The technique we use to "unto" the Chain Rule is called "substitution" or "change of variable". Consider the problem

$$\int (3x+2)^3\ dx$$

If we simply applied our power rule we would get

$$\int (3x+2)^3\ dx = \frac{1}{4}(3x+2)^4 + C$$

But, now if we differentiated our new antiderivative we would get

$$(3x+2)^3(3)$$

which is not our original problem. It's off by a factor of 3. Let's try the problem again by making a simple substitution and let $u = 3x + 2$. The plan is to rewrite the entire integral in terms of u instead of x. This means we will also need to write the dx in terms of u. Here's how it goes.

Let $u = 3x + 2$

Now find $\frac{du}{dx}$.

$\frac{du}{dx} = 3$ which, if we multiply each side by dx yields $du = 3\,dx$ or $\frac{1}{3}du = dx$.

The original problem can now be written as

$$\frac{1}{3}\int u\ du$$

The fraction is normally put outside the integral.

We can now antidifferentiate using the power rule.

$$\frac{1}{3}\int u\ du = \frac{1}{3}\frac{1}{4}u^4 + C = \frac{1}{12}u^4 + C$$

We now replace the original function.

$$\frac{1}{12}(3x+2)^4 + C$$

We must use this technique because all of our antidifferentiation theorems essentially only operate on single variables, not on expressions!

Example 1

Find $\displaystyle\int (7x-3)^8\ dx$.

$$\text{Let } u = 7x + 3 \ \longrightarrow\ \frac{1}{7}du = dx$$

$$\int (7x-3)^8\ dx = \frac{1}{7}\int u^8\ du = \frac{1}{7}\frac{1}{9}u^9 + C$$

$$\frac{1}{63}(7x+3)^9 + C$$

In the remaining examples, the work will be shown the way the problems are normally done–and the way you will show your work–on quizzes, tests and in your problem packets.

Example 2

Find $\displaystyle\int \sqrt{3x+4}\ dx$.

$$u = 3x + 4 \ \longrightarrow\ du = 3\ dx \ \longrightarrow\ \frac{1}{3}\ du = dx$$

$$\begin{aligned}\int \sqrt{3x+4}\ dx &= \frac{1}{3}\int u^{(1/2)}\ du\\ &= \frac{1}{3}\frac{2}{3}u^{(3/2)} + C\\ &= \frac{2}{9}\sqrt{(3x+4)^3} + C\end{aligned}$$

Example 3

Find $\int x^2(5+2x^3)^8\ dx$.

$$u = 5+2x^3 \longrightarrow du = 6x^2\ dx \longrightarrow \frac{1}{6}\ du = x^2dx$$

$$\begin{aligned}\int x^2(5+2x^3)^8\ dx &= \frac{1}{6}\int u^8\ du\\ &= \frac{1}{6}\frac{1}{9}u^9 + C\\ &= \frac{1}{54}(5+2x^3)^9 + C\end{aligned}$$

Example 4

Find $\int x\cos x^2\ dx$.

$$u = x^2 \longrightarrow du = 2x\ dx \longrightarrow \frac{1}{2}\ du = x\ dx$$

$$\begin{aligned}\int x\cos x^2\ dx &= \frac{1}{2}\int \cos u\ du\\ &= \frac{1}{2}\sin u + C\\ &= \frac{1}{2}\sin x^2 + C\end{aligned}$$

Example 5

Find $\int \frac{4x^2dx}{(1-8x^3)^4}\ dx$.

$$u = 1-8x^3 \longrightarrow du = -24x^2\ dx \longrightarrow -\frac{1}{6}\ du = 4x^2\ dx$$

(Note that we only divided by 6 because the original problem already has a 4 in the numerator. How constants are handled are up to you. . . just do what makes you most comfortable.)

$$\begin{aligned}\int \frac{4x^2dx}{(1-8x^3)^4}\ dx &= -\frac{1}{6}\int u^{-4}\ du \\ &= \left(-\frac{1}{6}\right)\left(-\frac{1}{3}\right)u^{-3}+C \\ &= \frac{1}{18}(1-8x^3)^{-3}+C \\ &= \frac{1}{18(1-8x^3)^3}+C\end{aligned}$$

Example 6

Find $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\ dx$.

$$u=\sqrt{x}\ \longrightarrow\ du=\frac{1}{2\sqrt{x}}\ dx\ \longrightarrow\ 2\ du=\frac{1}{\sqrt{x}}\ dx$$

$$\begin{aligned}\int \frac{\sin\sqrt{x}}{\sqrt{x}}\ dx &= 2\int \sin u\ du \\ &= -2\cos u + C \\ &= -2\cos\sqrt{x}+C\end{aligned}$$

Example 7

Find $\int \sin x\sqrt{1-\cos x}\ dx$.

$$u=1-\cos x\ \longrightarrow\ du=\sin x\ dx$$

$$\begin{aligned}\int \sin x\sqrt{1-\cos x}\ dx &= \int u^{1/2}\ du \\ &= \frac{2}{3}u^{3/2}+C \\ &= \frac{2}{3}\sqrt{(1-\cos x)^3}+C\end{aligned}$$

Example 8

Find $\int e^{5x}\ dx$.

$$u=5x\ \longrightarrow\ du=5\ dx\ \longrightarrow\ \frac{1}{5}\,du=dx$$

$$\begin{aligned}\int e^{5x}\ dx &= \frac{1}{5}\int e^u\ du\\ &= \frac{1}{5}\,e^u + C\\ &= \frac{1}{5}\,e^{5x} + C\end{aligned}$$

Note: Differentiating e raised to any linear quantity is easy… we've done it many times. The derivative of e^{8x-1} is just $8e^{8x-1}$. Antidifferentiating e raised to a linear quantity is equally simple. Let's derive a theorem!

Consider $\int e^{ax+b}\ dx$

$$u = ax + b \ \longrightarrow\ du = a\ dx \ \longrightarrow\ \frac{1}{a}\,du = dx$$

$$\begin{aligned}\int e^{ax+b}\ dx &= \frac{1}{a}\int e^u\ du\\ &= \frac{1}{a}\,e^u + C\\ &= \frac{1}{a}\,e^{ax+b} + C\end{aligned}$$

Now we can antidifferentiate e raised to a linear quantity without resorting to substitution!

Example 9

Find $\int e^{9x+4}\ dx$.

$$\int e^{9x+4}\ dx = \frac{1}{9}\,e^{9x+4}.$$

In fact, we will learn many "linear shortcuts" for antidifferentiating as we move on.

Example 10

Find $\int 2xe^{x^2}\ dx$.

The exponent in this case is not linear so we cannot use any shortcut!

$$u = x^2 \ \longrightarrow\ du = 2x\,dx$$

(We'll leave the 2 where it is since there is a 2 in our original problem.)

$$\begin{aligned}\int 2xe^{x^2}\ dx &= \int e^u\ du \\ &= e^u + C \\ &= e^{x^2} + C\end{aligned}$$

We now come to another type of substitution problem called "double substitution". This usually occurs when our choice for u is clear but the degree of this u is equal to or less than the degree of the the part of the expression that is not u.

Example 11

Find $\displaystyle\int x\sqrt{x+1}\ dx$.

$$\text{Let } u = x+1 \ \longrightarrow\ du = dx$$

This allows us to replace $x+1$ with u and the dx with du but leaves us with the x in front of the radical.

Since we know that $u = x+1$ then $x = u-1$. Now we can replace the x with a $u-1$.

$$\begin{aligned}\int x\sqrt{x+1}\ dx &= \int (u-1)\sqrt{u}\ du \\ &= \int u^{1/2}(u-1)\ du \\ &= \int \left(u^{3/2} - u^{1/2}\right)\ du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{5}\sqrt{(x+1)^5} - \frac{2}{3}\sqrt{(x+1)^3} + C\end{aligned}$$

Example 12

Find $\displaystyle\int x^2\sqrt{1+x}\ dx$.

$$u = 1+x \ \longrightarrow\ du = dx$$

$$x = u-1 \ \longrightarrow\ x^2 = u^2 - 2u + 1$$

$$\begin{aligned}\int x^2\sqrt{1+x}\ dx &= \int (u^2-2u+1)\sqrt{u}\ du\\ &= \int u^{1/2}(u^2-2u+1)\ du\\ &= \int \left(u^{5/2}-2u^{3/2}+u^{1/2}\right)\ du\\ &= \frac{2}{7}u^{7/2}-\frac{4}{5}u^{5/2}+\frac{2}{3}u^{3/2}+C\\ &= \frac{2}{7}\sqrt{(1+x)^7}-\frac{4}{5}\sqrt{(1+x)^5}+\frac{2}{3}\sqrt{(1+x)^3}+C\end{aligned}$$

Example 13

Find $\displaystyle\int (x+3)\sqrt{x-4}\ dx$.

$$u = x-4 \longrightarrow du = dx$$

$$x = u+4 \longrightarrow x+3 = u+7$$

$$\begin{aligned}\int (x+3)\sqrt{x-4}\ dx &= \int (u+7)u^{1/2}\ du\\ &= \int \left(u^{3/2}+7u^{1/2}\right)\ du\\ &= \frac{2}{5}u^{5/2}+\frac{14}{3}u^{3/2}+C\\ &= \frac{2}{5}\sqrt{(x-4)^5}+\frac{14}{3}\sqrt{(x-4)^3}+C\end{aligned}$$

7.2.3 Antidifferentiation theorems for trigonometric functions using substitution

Recall that we have theorems that allow us to take the antiderivative of sine and cosine. We do not yet have any theorems that allow us to antidifferentiate the other four trigonometric functions when they are "alone". For example, we have a theorem for example for antidifferentiating $\sec^2 x$, but none yet for just $\sec x$. We have a theorem for antidifferentiating $\sec x\tan x$ but not one for $\tan x$.

Let's start with $\tan x$.

Consider $\displaystyle\int \tan x\ dx$.

We know that $\tan x = \dfrac{\sin x}{\cos x}$. If we now use substitution with $u = \cos x$, we can derive a theorem for the antiderivative of $\tan x$!

$$u = \cos x \longrightarrow du = -\sin x dx \longrightarrow -du = \sin x dx$$

$$\begin{aligned}\int \tan x \, dx &= -\int \frac{1}{u} \, du \\ &= -\ln|u| + C \\ &= \ln\left|u^{-1}\right| + C \\ &= \ln\left|(\cos x)^{-1}\right| + C \\ &= \ln|\sec x| + C\end{aligned}$$

There. . . that wasn't so bad! We now have the following theorem.

$$D_x[\tan x] = \ln|\sec x| + C$$

Now let's look at $\cot x$.

Consider $\int \cot x \, dx$.

We know that $\cot x = \dfrac{\cos x}{\sin x}$. If we now use substitution with $u = \sin x$, we can derive a theorem for the antiderivative of $\cot x$!

$$u = \sin x \longrightarrow du = \cos x dx$$

$$\begin{aligned}\int \cot x \, dx &= \int \frac{\cos x}{\sin x} \, dx \\ &= \int \frac{1}{u} \, du \\ &= \ln|u| + C \\ &= \ln|\sin x| + C\end{aligned}$$

$$D_x[\cot x] = \ln|\sin x| + C$$

It's time for $\sec x$!

Consider $\int \sec x \, dx$.

This one requires something a little different. Remember that one technique for deriving theorems is to change the way the problem looks by adding zero or multiplying by one. This time we'll multiply by one. In this case, one will look like:

$$\frac{\sec x + \tan x}{\sec x + \tan x}.$$

$$\int \sec x \; dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \; dx$$
$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx$$

Now, $u = \sec x + \tan x \;\longrightarrow\; du = (\sec x \tan x + \sec^2 x)\, dx$

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx = \int \frac{1}{u} \; du$$
$$= \ln |u| + C$$
$$= \ln |\sec x + \tan x| + C$$

$$D_x[\sec x] = \ln |\sec x + \tan x| + C$$

The antiderivative of $\csc x$ is derived in much the same way.

$$D_x[\csc x] = \ln |\csc x - \cot x| + C$$

There... now we've got antiderivatives for all six trigonometric functions! (Time to memorize!!)

7.3 Differential Equations

7.3.1 Introduction

This is one of the most important topics we learn about all year. The people to create the AP Calculus test love differential equations. Most years differential equations are scattered throughout the multiple choice and almost every year you will have to solve a differential equation in the free response. So take note!

A differential equation is simply an equation that has derivatives in it. In fact, when you find a derivative, your answer is a differential equation! Let's say you were given

$$y = x^2$$

and were asked to find the derivative. Of course you would write

$$\frac{dy}{dx} = 2x.$$

This is a differential equation! It's an equation with a derivative on the left side of the equation.

Solving differential equations is an extension of our work in antidifferentiating. There are many types of differential equations and many techniques that can be used to solve them. In fact, at the university level, there are entire courses dedicated to solving differential equations. The course takes after the calculus sequence is normally one called "Differential Equations"... or more commonly among students, "Diffy Q".

There are actually many different types of differential equations. Fortunately for us, we will deal with only the simplest... first-order, separable differential equations. "First-order" means that the highest derivative in the problem is a first derivative. (We may look a few very simple second-order examples.) "Separable" means that we can separate the variables, getting all the y terms on one side and all the x terms on the other.

The *general solution* to a differential equation is a family of functions. A *particular solution* is one member of that family that passes through a particular point.

7.3.2 Solving differential equations

To solve a differential equation we first need to separate the variables, then antidifferentiate each side. Many times we will be given an "initial condition", a point through which our curve passes. We can use this initial condition to find the C we will get as a result of antidifferentiation.

Example 1

Solve $\frac{dy}{dx} = 2x$.

$$\begin{aligned} \frac{dy}{dx} &= 2x \\ dy &= 2xdx \\ \int dy &= \int 2x\ dx \\ y &= x^2 + C \end{aligned}$$

Now, there are several things to note here. We antidifferentiated both sides but we have only one C. What we actually did was to get a constant on both sides and then combined them onto one side... they're just constants. We were not given any initial condition so this is a "general solution" as opposed to a "particular solution".

Example 2

Solve $\frac{dy}{dx} = 2x$ if $y = 7$ when $x = 2$.

This is almost the same problem as Example 1. The only difference is that we need to find the one member of the family of functions $y = x^2 + C$ that passes through the point $(2, 7)$.

$$\frac{dy}{dx} = 2x$$
$$dy = 2xdx$$
$$\int dy = \int 2x \; dx$$
$$y = x^2 + C$$

Substituting the given x and y yields $C = 3$.

Therefore the solution is

$$y = x^3 + 3$$

Example 3

Given $\frac{dy}{dx} = \frac{2x^2}{3y^2}$, find $y = f(x)$.

Important: Note that the problem asks us to find $y = f(x)$. This means that when we find our solution, it must be solved for y.
It is also fairly common to skip the step where we show the antidifferentiation symbol as you will see in the next example.

$$\frac{dy}{dx} = \frac{2x^2}{3y^2}$$
$$3y^2dy = 2x^2dx$$
$$y^3 = \frac{2}{3}x^3 + C$$
$$y = \sqrt[3]{\frac{2}{3}x^3 + C}$$

Example 4

Given $\frac{dy}{dx} = \frac{x+1}{xy}$ and $y = -4$ when $x = 1$, find $y = f(x)$.

$$\frac{dy}{dx} = \frac{x+1}{xy}$$
$$y \; dy = \frac{x+1}{x} \; dx$$
$$y \; dy = 1 + \frac{1}{x} \; dx$$
$$\frac{1}{2}y^2 = x + \ln|x| + C$$

$$\text{Since } y = -4 \text{ when } x = 1 \longrightarrow C = 7$$

$$\begin{aligned} \frac{1}{2}\,y^2 &= x + \ln|x| + 7 \\ y^2 &= 2x + 2\ln|x| + 14 \\ y &= \pm\sqrt{2x + 2\ln|x| + 14} \end{aligned}$$

Since the initial conditions specify that $y = -4$, we choose the negative root.

Therefore $y = -\sqrt{2x + 2\ln|x| + 14}$.

Example 5

Find the general solution: $\dfrac{dy}{dx} = e^{u+2t}$.

$$\begin{aligned} \frac{dy}{dx} &= e^{u+2t} \\ \frac{dy}{dx} &= e^u\ e^{2t} \\ \frac{1}{e^u}\ du &= e^{2t}dt \\ e^{-u}\ du &= e^{2t}dt \\ -e^{-u} &= \frac{1}{2}\ e^{2t} + C \\ \frac{2}{e^u} &= -e^{2t} + D \end{aligned}$$

Example 6

Find the general solution: $\dfrac{dy}{dx} = \dfrac{\ln x}{xy + xy^3}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\ln x}{xy + xy^3} \\ \left(xy + xy^3\right) dy &= \ln x\ dx \\ \left(y + y^3\right) dy &= \frac{\ln x}{x}\ dx \end{aligned}$$

The left side is easy...but the right side requires substitution where $u = \ln x \longrightarrow du = \dfrac{1}{x}\ dx$

The right side now becomes $\displaystyle\int u\ du$ which in turn becomes $\dfrac{1}{2}u^2 + C$ where the u is $\ln x$.

$$\begin{aligned} \frac{1}{2}y^2 + \frac{1}{4}y^4 &= \frac{1}{2}\ln^2 x + C \\ 2y^2 + y^4 &= 2\ln^2 x + D \end{aligned}$$

Example 7

Find the general solution: $\dfrac{d^2y}{dx^2} = 4x + 3.$

This is a second-order differential equation. The symbol $\dfrac{d^2y}{dx^2}$ cannot be separated like $\dfrac{dy}{dx}$. We can "get around" this with a little symbol manipulation. At times the first derivative is written as y', so the derivative of the derivative is $\dfrac{dy'}{dx}$. This is a fraction and can be separated.

$$\begin{aligned}
\frac{d^2y}{dx^2} &= 4x + 3 \\
\frac{dy'}{dx} &= 4x + 3 \\
dy' &= (4x + 3)\, dx \\
y' &= 2x^2 + 3x + C \\
\frac{dy}{dx} &= 2x^2 + 3x + C \\
dy &= (2x^2 + 3x + C)\, dx \\
y &= \frac{2}{3}\, x^3 + \frac{3}{2}\, x^2 + Cx + D
\end{aligned}$$

At times we will be given a set of initial conditions like $y' = -3$ and $y = 2$ when $x = 1$. In this case we would stop once we got y' and use the initial condition $y' = -3$ when $x = 1$ to find the C. Then use the other initial condition, $y = 2$ when $x = 1$, to find D.

7.3.3 Differential equations and the exponential growth model

If we had known how to solve differential equations, the derivation of the exponential growth model we did previously would have made much more sense!

We started with

$$\frac{dy}{dx} = ky.$$

We then brought the y to the left side and wrote

$$\frac{1}{y}\,\frac{dy}{dx} = k.$$

We then used "reverse implicit differentiation" (not a real term!) to find a function whose derivative is

$$\frac{1}{y}\,\frac{dy}{dx}.$$

It would have been much easier to simply separate the variables and write

$$\frac{1}{y}\, dy = k\, dt.$$

The next step would be

$$\ln |y| = kt + C$$

followed by

$$y = e^{kt+C}$$

then

$$y = e^{kt}e^{C}$$

and finally

$$y = Ae^{kt}.$$

7.4 Slope Fields

7.4.1 Introduction

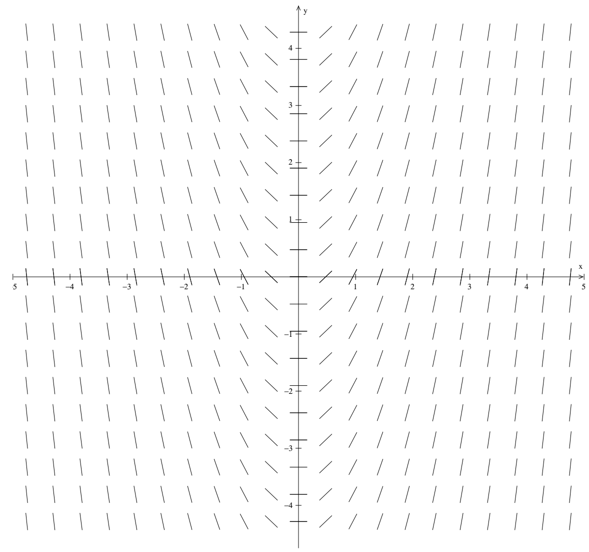
Slope fields are used to visualize solutions to differential equations. We use them to visualize solutions–even if the solution is not available to us using analytic tools! That's why they are important. When you look at a slope field, you are not looking at the graph of a derivative but rather the solution to a differential equation.

We need to be able to perform four tasks with slope fields.

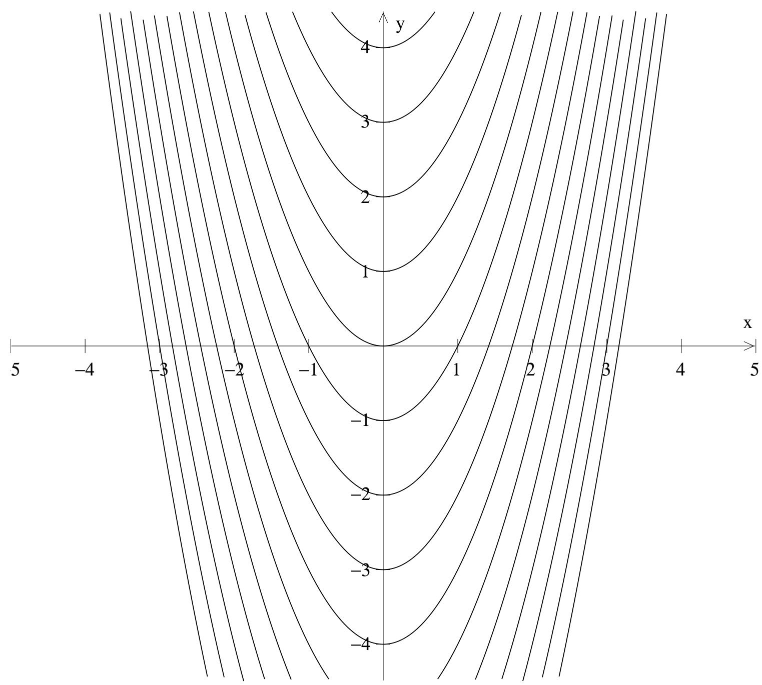
- Draw a simple slope field.
- Match a slope field with a differential equation.
- Match a slope field with the solution to a differential equation.
- On a given slope field, draw a sketch of a particular solution given a set of initial conditions.

Consider the differential equation $\dfrac{dy}{dx} = 2x$. Solving this equation yields the general solution $y = x^2 + C$. The slope of a tangent to any member of the family of curves $y = x^2 + C$ can be found by taking the x-coordinate and multiplying by two. If, for example, we began with the initial condition that $y = 5$ when $x = 1$, then $C = 4$ and the particular solution is $y = x^2 + 4$. Now, the slope at say, $x = 7$ would be 14.

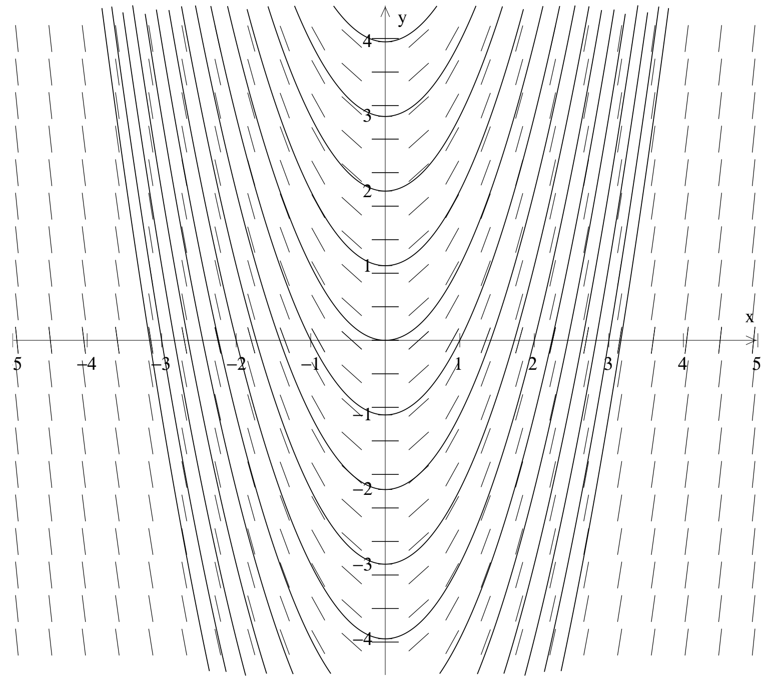
The slope field for $\dfrac{dy}{dx} = 2x$ is shown below.



"Wonderful!" you say, "What is it?". All those little line segments are actually short tangent lines. Let's back up for a moment and look at the graph of several members of the family of curves $y = x^2 + C$.



Now, let's combine the two graphs.



When we look at a slope field, we are looking at segments of tangents to members of the family of curves which is the solution to a differential equation. It's as if we sketched dozens of members of a family of curves, then went to selected points on the coordinate grid and drew short tangents to each of the curves. . . and then erased the actual family of curves and left the tangent segments!

7.4.2 Matching slope fields and differential equations

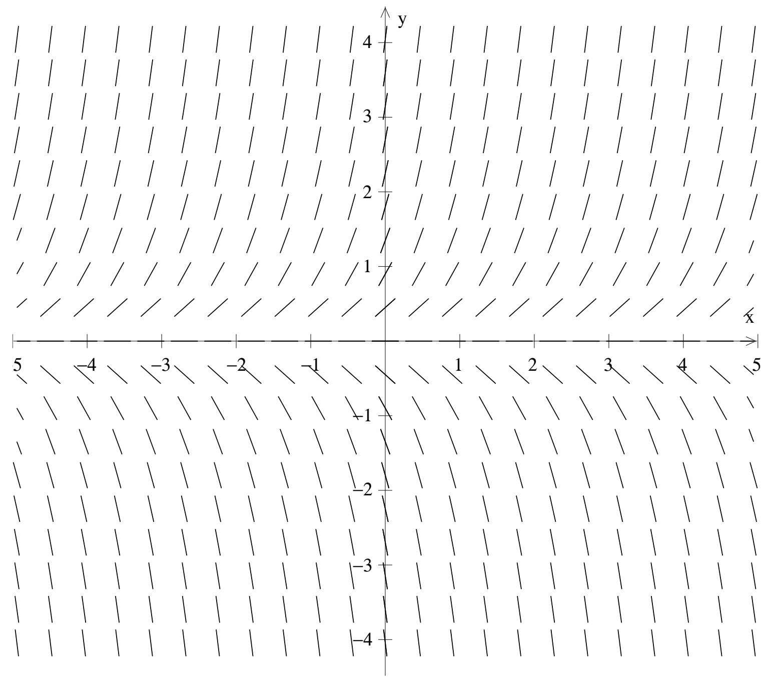
Now, how can we match a slope field to a differential equation. One way is to look at the differential equation, pick any point on the grid, calculate what the slope of a tangent should be and look to see if the calculated slope matches the slope on the given slope field. For instance, if our differential equation is

$$\frac{dy}{dx} = \frac{x-1}{y}$$

we should see horizontal segments at all points where $x = 1$ because if $x = 1$, the slope will always be zero. Except, of course, at $(1, 0)$ where we would have a vertical tangent because $\frac{x-1}{y}$ at $(1, 0)$ does not exist.

Another way to match a differential equation to a slope field is to look at the differential equation itself. Note that in our example $\frac{dy}{dx} = 2x$, the differential equation contains no y's and our slope field consisted of columns of parallel segments! If the differential equation had only y's in it (no x's) we would see rows of parallel segments.

Below is the slope field for $\frac{dy}{dx} = 2y$.



Note that there are rows of parallel segments.

If the differential equation contains both x and y, it is more difficult to match it with a slope field. In this case we can either try to solve the differential equation and look for a slope field that looks like members of that family (we'll practice this in class) or better yet, calculate the slope of a tangent at selected points as described above.

7.4.3 Drawing a slope field

Let's now walk through the process of drawing a slope field. Consider the differential equation

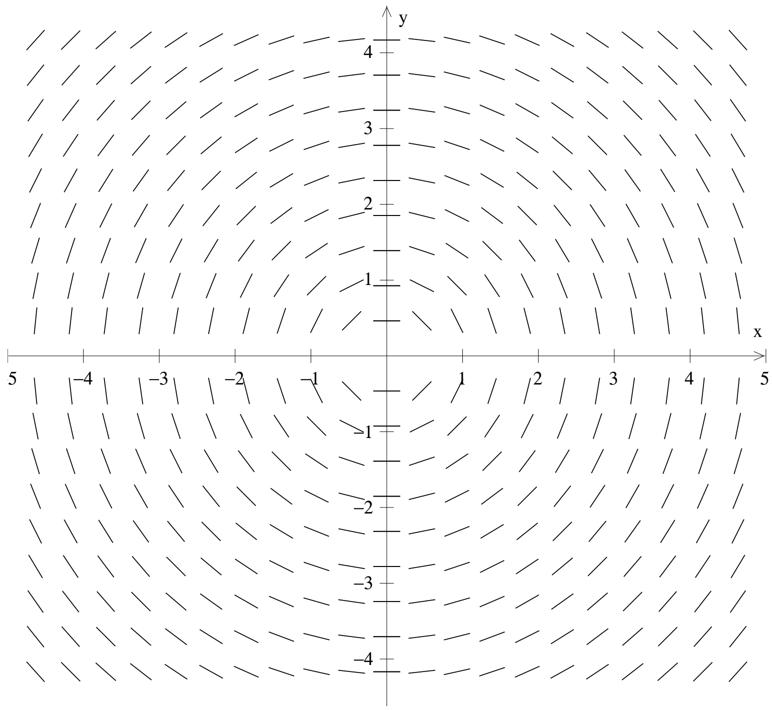
$$\frac{dy}{dx} = -\frac{x}{y}$$

In this example, the slope of a tangent to any member of the solution family depends on both x and y. Begin by selecting points on the grid and plugging in the values of x and y into

$$-\frac{x}{y}.$$

Let's say you chose the point $(5, 5)$. The slope of a tangent would then be 1. Go the the point $(5, 5)$ on the grid and draw a short line segment that has a slope of 1. Repeat this process for as many points as required. (Most of the time you will be given a grid and normally have to draw at most twelve segments or so.) Note that any point where $x = 0$ you will have horizontal tangents–along the y-axis. Any point where $y = 0$ you will have vertical tangents. Many times vertical tangents are not drawn–simply do not draw a segment at these points.

The slope field for this particular solution is shown below.

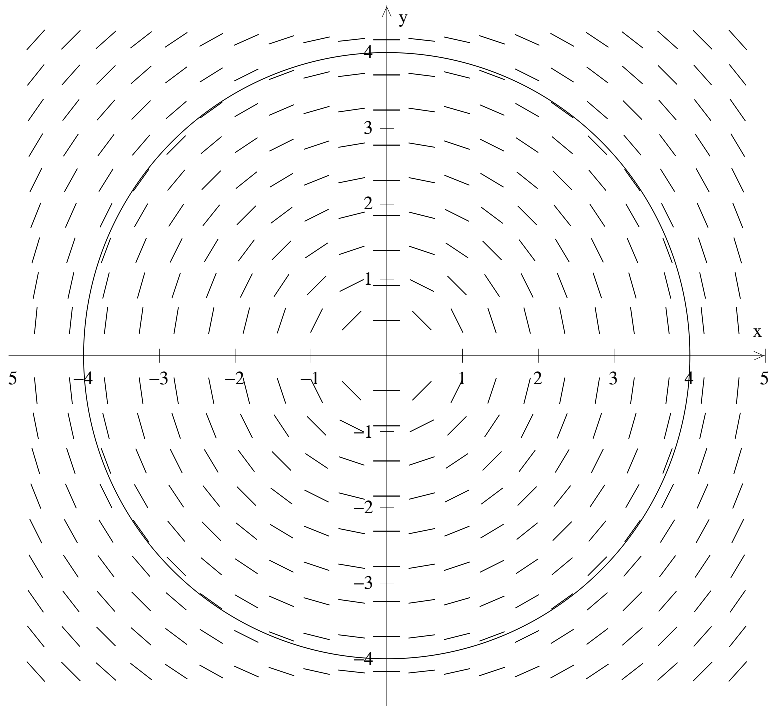


It would appear that the solution is the family of circles centered at the origin.

7.4.4 Drawing a particular solution on a slope field

Now, if we have to draw a particular solution we start at the point indicated by the initial condition. Let's say the initial condition is that $y = 0$ when $x = 4$. Start at the point (4.0) and "follow the flow" of the segments (officially inflectional tangents). Your particular solution should never cross over any of the segments but instead should flow between them.

The particular solution with the initial condition $y = 0$ when $x = 4$ is shown below.



We have now covered the four tasks we need to perform with slope fields:

- Draw a simple slope field.
- Match a slope field with a differential equation.
- Match a slope field with the solution to a differential equation.
- On a given slope field, draw a sketch of a particular solution given a set of initial conditions.

Of course, we will go over many more examples in class.

Chapter 8

The Definite Integral

8.1 Introduction

The definite integral is one of the most important and fascinating concepts in Calculus. Like the derivative, it is a limit. The derivative is a limit of a quotient–the definite integral is a limit of a sum. We've got some preliminary work to do before we actually define the definite integral and learn its many applications. Our first step will be to review summation notation.

8.2 Summation–or Sigma Notation

To see how the notation works, consider the following sum.

$$1^2 + 2^2 + 3^2 + 4^2 = 5^2$$

Each term is of the form of i^2. The sum begins with $i = 1$ and ends with $i = 5$, with the i taking on all *integer* values from 1 to 5. There is no magic about the i. It is what we call a "dummy variable" and any letter can be used. Normally, it is convention to use i, j or k when using sigma notation.

Using sigma notation the sum given above is written

$$\sum_{i=1}^{5} i^2$$

The number below the sigma is called the "lower bound" and the number above the sigma is the "upper bound".

What not just write out the terms? It seems pretty easy with the example we just used but consider the sum all all the squares from 1 to 1000! Better have some time on your hands, that a 1000-term sum! Using sigma notion the same sum can be compactly expressed.

$$\sum_{i=1}^{1000} i^2$$

The notation becomes even more valuable with expression like the sum of all the $7i - i^3$ from -3 to 17, which can be written

$$\sum_{i=-3}^{17} \left(7i - i^3\right).$$

Note the parenthese... without them we would only be summing all the $7i$'s! Use of parentheses is critical.

Here are several more examples of sigma notation:

$$\sum_{i=-2}^{1} (3i+2) = [3(-2)+2] + [3(-1)+2] + [3(0)+2] + [3(1)+2]$$

$$\sum_{i=1}^{4} f(i) = f(1) + f(2) + f(3) + f(4)$$

$$\sum_{i=1}^{4} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

There are times when we do not wish to specify an upper bound on a sum. In this case, we use n as the upper bound. Here are several examples:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 363 + \ldots + (n-2)^3 + (n-1)^3 + n^3$$

$$\sum_{i=1}^{n} A_i = A_1 + A_2 + A_3 + \ldots + A_{n-2} + A_{n-1} + A_n$$

And one more very important example...

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_{n-2})\Delta x + f(x_{n-1})\Delta x + f(x_n)\Delta x$$

8.2.1 Summation theorems

We will now present several theorems involving summations. They will be presented with proof because you no doubt saw them all proven last year... this is just a reminder.

The sum of a constant:

$$\sum_{i}^{n} c = cn$$

The sum of a constant is the product of the constant and the upper bound. . . as long as the lower bound is one. For example:

$$\sum_{i=1}^{5} 8 = (8)(5) = 40$$

All of the summation theorems have a lower bound of one. We will soon address a theorem that will allow is to adjust the bounds on a sum in order to use our theorems.

The sum of a function multiplied by a constant:

$$\sum_{i=1}^{n} cf(x) = c \sum_{i=1}^{n} f(x)$$

Note that like derivatives and antiderivatives, constants are dealt with differently depending on whether or not they are stand-alone constants or if they are coefficients.

We will now look at four theorems which you need to memorize.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

The sum of all the numbers from 1 to 100 can be written and evaluated as follows:

$$\sum_{i=1}^{100} i = \frac{100(101)}{2} = 5050$$

The sum of all the numbers from 1 to 100 has a little history behind it. . . and maybe a little legend but it's still a good story. Karl Friedrich Gauss, perhaps the greatest mathematician to have ever stomped upon the terra, was a child prodigy. As a young student, Gauss had a teacher who probably never won the "Best Liked Teacher" award (although later he recognized Karl's talent and arranged a special teacher for him). As a punishment one day, the boys in the class were instructed that before they left for the day, they had to all all the integers from one to one-hundred. Now, the boys didn't have paper and pencil or calculators. . . all they had were slates and pieces of crude chalk. Evidently, the teacher calculated the sum previously and watched with amusement as the boys began their calculations. After just a few moments, Gauss wrote the answer on his slate and turned it in. The teacher was sure that Gauss had the incorrect answer after such a short time. . . but he was wrong! Gauss was allowed to leave. It turns out that Gauss didn't actually add up each individual integer and did not use the summation theorem discussed above. He started to write the numbers down and noticed a pattern. The looks like this: $1+2+3\ldots+98+99+100$. Young Karl noticed that if he added the first and last terms the total was 101. Then he added the second and the second to last term and got 101 again. Well, there would be fifty such pairs so $50(101) = 5050$. Notice that the $50(101) = 50(100+1) = \frac{100(100+1)}{2}$, exactly what you get if you applied the above theorem directly. By the way, there are many interesting stories about Gauss. If you'd like to read some, let me know.

Time for two more theorems to memorize.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Notice that the sum of the i^3's from 1 to n is the square of the sum of the i's from 1 to n. The expansion of the sum of all the i's and i^2's are given because we use them so much later on.

All of the theorems listed so far have all had a lower bound of one. What happens when we need to sum a large number of terms and the sum does not begin at one. Consider the following two sum.

$$\sum_{i=5}^{9} i = 5+6+7+8+9$$

This sum is the same as the following sum.

$$\sum_{i=1}^{5} i+4 = 5+6+7+8+9$$

Subtracting four from each bound and then adding four to the argument results in the same sum! Here's another example. The following sum

$$\sum_{i=3}^{7} i^2 = 3^2+4^2+5^2+6^2+7^2$$

is the same as

$$\sum_{i=1}^{5} (i+2)^2 = 3^2+4^2+5^2+6^2+7^2$$

The same process works if we need to add to the bounds. In this example we will add three to each bound and subtract three from the argument.

$$\sum_{i=-2}^{1} 3^i = 3^{-2}+3^{-1}+3^0+3^1$$

is the same as

$$\sum_{i=1}^{4} 3^{i-3} = 3^{-2} + 3^{-1} + 3^{0} + 3^{1}$$

Here's the theorem.

$$\sum_{i=a}^{b} f(i) = \sum_{i=a+c}^{b+c} f(i-c)$$

and

$$\sum_{i=a}^{b} f(i) = \sum_{i=a-c}^{b-c} f(i+c)$$

Example 1

Evaluate $\sum_{i=3}^{5} (4i+3)$.

Note that the lower and upper bounds are fairly close–there will be only three terms to sum.In cases like this, do not use any theorems, just write out the terms and add.

$$\begin{aligned}\sum_{i=3}^{5} (4i+3) &= (12+3) + (16+3) + (20+3) \\ &= 57\end{aligned}$$

Example 2

Evaluate $\sum_{i=1}^{20} (5i+4)$.

Unlike Example 1, this sum will have a large number of terms. The best approach for this type of problem is to do the general sum first. . . from $i = 1$ to n. Then evaluate this expression for $n = 20$.

Consider $\sum_{i=1}^{n} (5i+4)$

$$\sum_{i=1}^{n} (5i+4) = 5\,\frac{n(n+1)}{2} + 4n$$

For $n = 20$,

$$\sum_{i=1}^{20}(5i+4) = 5\,\frac{20(20+1)}{2} + 4(20) = 1130$$

Example 3

Evaluate $\displaystyle\sum_{i=1}^{20} i(3i-2)$.

$$\text{Consider } \sum_{i=1}^{n} i(3i-2)$$

$$\begin{aligned}\sum_{i=1}^{n} i(3i-2) &= \sum_{i=1}^{n}(3i^2-2i)\\ &= 3\,\sum_{i=1}^{n} i^2 - 2\,\sum_{i=1}^{n} i\\ &= 3\,\frac{n(n+1)(2n+1)}{6} - 2\,\frac{n(n+1)}{2}\end{aligned}$$

For $n = 20$

$$\sum_{i=1}^{20} i(3i-2) = 3\,\frac{20(20+1)(2(20)+1)}{6} - 2\,\frac{20(20+1)}{2} = 8190$$

Notice that in our first step, we distributed the i before we used any summation theorems. We needed to do this because even though the summation of a sum is equal to the sum of the summations, the summation of a product is *not* the product of the summations! We ran into this same situation with derivatives and antiderivatives. Sums of terms are treated very differently than products of terms.

Example 4

Evaluate $\displaystyle\sum_{i=3}^{6} \frac{2}{i(i-2)}$.

Not only do we not have any theorems for fractions, this sum will involve only four terms so it's best just to write out the terms.

$$\begin{aligned}\sum_{i=3}^{6}\frac{2}{i(i-2)} &= \frac{2}{3(1)} + \frac{2}{4(2)} + \frac{2}{5(3)} + \frac{2}{6(4)}\\ &= \frac{17}{15}\end{aligned}$$

Example 5 (very important example)

There will be many times when we want a general expression for a sum. The ability to do this will be *critical* when we start working with the definition of the definite integral.

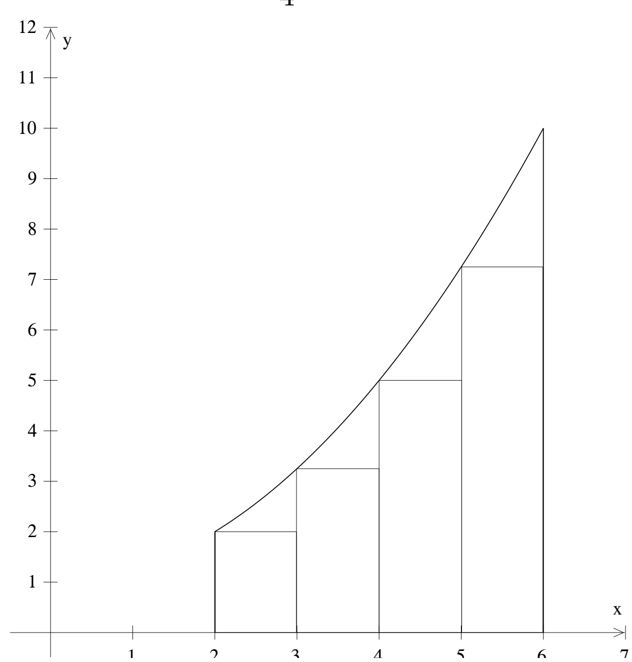
Expand and simplify $\sum_{i=1}^{n}(3i^2+5i)$.

$$\begin{aligned}\sum_{i=1}^{n}(3i^2+5i) &= 3\,\frac{2n^3+3n^2+n}{6}+5\,\frac{n^2+n}{2}\\ &= \frac{2n^3+3n^2+n}{2}+\frac{5n^2+5n}{2}\\ &= \frac{2n^3+8n^2+6n}{2}\\ &= n^3+4n^2+3n\end{aligned}$$

8.3 Area Under a Curve–Approximations

8.3.1 Introduction

The method we will describe here to estimate the area under a curve is an old one. In fact, it is the same method employed by Archimedes. The basic idea is simple. Since we cannot yet find the exact area under a curve, we will estimate the area using rectangles. (By "area under a curve" we mean the area bounded by the curve and the x-axis.)The graph below illustrates a problem in which four rectangles of equal width are being used to estimate the area under $f(x)=\frac{1}{4}x^2+1$ between $x=2$ and $x=6$.



To estimate the area, we add up the areas of the four rectangles. The more rectangles we use, the more accurate our estimate will be. Notice that in this case, no matter how many rectangles we use, there will always be small regions on the top of each each rectangle. By increasing the number of rectangles we can "exhaust" these small areas. That is why this technique is called area by "exhaustion". There are many ways for the rectangles to be drawn. In the above diagram we used what is called a left sum. We can also do right sums and midpoint sums. In the past (but no longer thankfully) we also used inscribed rectangles (in which the rectangles always lied within the curve) and circumscribed rectangles (in which part of each rectangle was outside of the region.)

8.3.2 The details

The first step in this process is to "partition the interval" into equal subintervals. To do this you divide the width of the region by the number of rectangles. In the diagram above we partitioned the interval from $x = 2$ to $x = 6$ into four subintervals. We then used the left side of each subinterval to create our rectangles.

Let's run through another example from beginning to end.

We will estimate the area of the region bounded by $f(x) = -x^2 + 9$ from $x = 0$ to $x = 2$ using four subintervals of equal width and a left sum.

First partition the interval. To partition the interval, we divide the interval into equal subintervals. We partition any interval $[a, b]$ into n subintervals by dividing the length of the interval by n. Each subinterval is then Δx wide. So each subinterval can be found by

$$\Delta x = \frac{b-a}{n}.$$

The left end of the interval is denoted x_0 and the right end is denoted x_n. Thus, on the interval $[a, b]$, $x_0 = a$ and $x_n = b$. In general, to partition an interval $[a, b]$ into n subintervals we say:

$$\begin{aligned}
x_0 &= a \\
x_1 &= a + \Delta x \\
x_2 &= a + 2\Delta x \\
x_3 &= a + 3\Delta x \\
&\vdots \\
x_{n-2} &= a + (n-2)\Delta x \\
x_{n-1} &= a + (n-1)\Delta x \\
x_n &= b
\end{aligned}$$

This process may seem over complicated but it will be important when we begin our work on finding *exact* areas via a limit.

Now, when we are approximating areas, the reality of this process is much easier than it may look. Back to our problem.

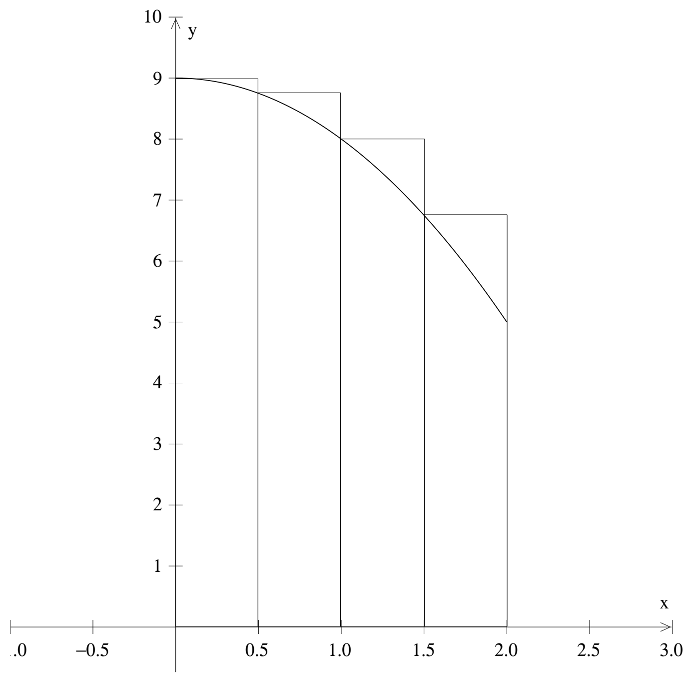
To partition the interval $[0, 2]$ into four subintervals of equal width we first find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

We know that $x_0 = a$ so $x_0 = 0$. We get x_1 by adding our Δx to x_0. We get x_2 by adding $Deltax$ to x_1 and so on until we get to x_n which in our case is x_4. This gives us the following.

$$\begin{aligned} x_0 &= 0 \\ x_1 &= \frac{1}{2} \\ x_2 &= 1 \\ x_3 &= \frac{3}{2} \\ x_4 &= 2 \end{aligned}$$

Our problem asks us to perform a left sum so we will use the left side of each subinterval to generate the heights of our rectangles. The diagram below shows the partition and the left-sum rectangles.



Again note that because this is a left sum, we will not use the right endpoint. $f(0)$ will be the height of the first rectangle, $f(.5)$ the height of the second and so on. We are now ready to estimate the area.

$$\begin{aligned} A &\approx \sum_{i=0}^{3} f(x_i)\Delta x \\ &\approx \Delta x\left[f(x_0) + f(x_1) + f(x_2) + f(x_3)\right] \\ &\approx \Delta x\left[f(0) + f(.5) + f(1) + f(1.5)\right] \\ &\approx \frac{1}{2}(9 + 8.750 + 8 + 6.750) \\ &\approx 16.250 \end{aligned}$$

We will now do a right sum without all the explanation and you'll see how straightforward the process really is.

Example 1 (right sum)

Estimate the area bounded by $f(x) = x^2 + 5x + 6$ and the x-axis from $x = -1$ to $x = 5$ using 5 subintervals of equal width and a right sum.

$$\Delta x = \frac{b-a}{n} = \frac{5 - -1}{5} = 1.200$$

$$\begin{aligned}
x_0 &= -1 \\
x_1 &= .200 \\
x_2 &= 1.400 \\
x_3 &= 2.600 \\
x_4 &= 3.800 \\
x_5 &= 5
\end{aligned}$$

$$\begin{aligned}
A &\approx \sum_{i=1}^{5} f(x_i)\Delta x \\
&\approx \Delta x \left[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) \right] \\
&\approx 1.200(7.040 + 14.960 + 25.760 + 39.440 + 56) \\
&\approx 171.840
\end{aligned}$$

Example 2 (left sum)

Estimate the area bounded by $f(x) = x^2 + 5x + 6$ and the x-axis from $x = -1$ to $x = 5$ using 5 subintervals of equal width and a right sum.

$$\Delta x = \frac{b-a}{n} = \frac{5 - -1}{5} = 1.200$$

$$\begin{aligned}
x_0 &= -1 \\
x_1 &= .200 \\
x_2 &= 1.400 \\
x_3 &= 2.600 \\
x_4 &= 3.800 \\
x_5 &= 5
\end{aligned}$$

$$\begin{aligned}
A &\approx \sum_{i=0}^{4} f(x_i)\Delta x \\
&\approx \Delta x \left[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) \right] \\
&\approx 1.200(2 + 7.040 + 14.960 + 25.760 + 39.440) \\
&\approx 107.040
\end{aligned}$$

The next example shows how to perform a midpoint sum. The only difference is that instead of using left or right endpoints of each subinterval to generate heights of rectangles, we will find the midpoint of each subinterval and use those midpoints to generate our rectangle heights.

Example 3 (midpoint sum)

Estimate the area bounded by $f(x) = x^2+5x+6$ and the x-axis from $x = -1$ to $x = 5$ using 5 subintervals of equal width and a right sum.

$$\Delta x = \frac{b-a}{n} = \frac{5 - -1}{5} = 1.200$$

$$\begin{aligned} x_0 &= -1 \\ x_1 &= .200 \\ x_2 &= 1.400 \\ x_3 &= 2.600 \\ x_4 &= 3.800 \\ x_5 &= 5 \end{aligned}$$

Now to find the midpoint of each subinterval, we can use the midpoint formula on the endpoints of each subinterval or we can find the midpoint of the first subinterval and then just add our Δx four times–which is what we will do.

$$\begin{aligned} m_1 = \frac{x_0 + x_1}{2} &= -.400 \\ m_2 &= .800 \\ m_3 &= 2 \\ m_4 &= 3.200 \\ m_5 &= 4.400 \end{aligned}$$

$$\begin{aligned} A &\approx \sum_{i=1}^{5} f(m_i)\Delta x \\ &\approx \Delta x \left[f(m_1) + f(m_2) + f(m_3) + f(m_4) + f(m_5)\right] \\ &\approx 1.200(4.160 + 10.640 + 20 + 32.240 + 47.360) \\ &\approx 137.280 \end{aligned}$$

Of course, the left, right and midpoint sums will give us different approximations. One skill we need to have is to list approximations in order. For discussion purposes, we will use L_s to denote a left sum, R_s to denote a right sum and M_s to denote a midpoint sum. For a function that is always decreasing on an interval, $R_s < M_s < L_s$. For functions that are always increasing, the relationship is reversed.

8.4 Exact Area Under a Curve

8.4.1 Introduction

We all realize that the more rectangles we use, the more accurate our area approximations are. In order to get the *exact* area, it's pretty clear that we will need to use an infinite number of rectangles. So how is it possible to add up an infinite number of rectangles? You already know . . . it involves limits!

8.4.2 The summation

When we approximated with a right sum, we did not use the left endpoint and so expressed the sum as:

$$\sum_{i=1}^{n} f(x_i)$$

When we approximated with a left sum, we did not use the right endpoint and so expressed the sum as:

$$\sum_{i=0}^{n-1} f(x_i)$$

Now, recall that all our theorems that allowed us to work with summations all went from $i = 1$ to n. Well, the right sum seems to fit right in but the left sum needs to be rewritten. You should also remember that there was a theorem that allowed us to change the upper and lower bound on a summation.

$$\sum_{i=a}^{b} f(i) = \sum_{i=a+c}^{b+c} f(i-c)$$

If we apply this theorem with $a = 1$ to our left sum we get

$$\sum_{i=0}^{n-1} f(x_i) = \sum_{i=1}^{n} f(x_{i-1})$$

At least now both sums go from 1 to n. If we need to do a left sum we use

$$\sum_{i=1}^{n} f(x_{i-1})$$

and if we need to do a right sum we use

$$\sum_{i=1}^{n} f(x_i)$$

Here's the thing, mathematicians hate having two expressions when one will do so here's what we do. We rewrite the sum as

$$\sum_{i=1}^{n} f(c_i)$$

where the $c_i = x_i$ for a right sum and $c_i = x_{i-1}$.

Alright, so far so good. We now have one expression we can use for left or right sums.

8.4.3 Partitioning an interval–one more time

As you recall from our last section, we begin partitioning an interval by first finding the appropriate $Deltax$. The left endpoint of the interval is denoted x_0 and the right endpoint is denoted x_n. Each partition is the labeled: $x_0,\ x_1,\ x_2,\ x_3,\ \ldots,\ x_{n-2},\ x_{n-1},\ x_n$. These are only labels–not real numbers. When we did approximations we had an interval like $[1, 5]$. This time we want to partition the general interval $[a, b]$.

The left end of this hypothetical interval is a so $x_0 = a$. The right end is b so $x_n = b$. Now, somewhere inside the interval there is something we call the ith subinterval. The left side of this subinterval is x_{i-1} and the right side is x_i. So, when we start adding Δx the partition looks like this:

$$\begin{gathered} x_0 = a \\ x_1 = a + \Delta x \\ x_2 = a + 2\Delta x \\ x_3 = a + 3\Delta x \\ \vdots \\ x_{i-1} = a + (i-1)\Delta x \\ x_i = a + i\ \Delta x \\ \vdots \\ x_{n-2} = a + (n-2)\Delta x \\ x_{n-1} = a + (n-1)\Delta x \\ x_n = b \end{gathered}$$

8.4.4 Exact area

Important note: When we find an approximate area by adding up rectangles we are performing something called a “Riemann Sum”. We need to add up an infinite number of rectangles–so we are finding areas via a “limit of a Riemann sum”.

Now we’re ready to find exact area. Keep in mind that $f(x_i)$ is simply the height of a rectangle and Δx is its width. We know that the more rectangles we use, the better our estimate of the area. If we want the exact area, we simply let the number of rectangles we use (the number of subintervals) to to infinity!

Exact area using a left sum: $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\ \Delta x$

Exact area using a right sum: $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$

where $x_i = a + i\ \Delta x$

and $x_{i-1} = a + (i-1)\Delta x$

As usual, we don’t like having two expressions so we normally write the limit of a Riemann sum to find exact area as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\ \Delta x$$

where $c_i = a + i\ \Delta x$ for right sums and

and $c_i = a + (i-1)\Delta x$ for left sums.

As you will soon realize, right sums are always easier!

Example 1

Find the exact area of the region bounded by $y = 3x$, $x = 3$, $x = 6$, and the x-axis using a right sum.

Since we are using a right sum we will use

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

where $x_i = 3 + i\ \Delta x$ and $\Delta x = \dfrac{6-3}{n} = \dfrac{3}{n}$

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(3 + i\ \Delta x)\ \Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} [3(3 + i\ \Delta x)]\ \Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} [9 + 3i\ \Delta x]\ \Delta x \end{aligned}$$

Now substitute $\Delta x = \dfrac{3}{n}$

$$\begin{aligned} &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[9 + 3i\ \frac{3}{n}\right]\ \frac{3}{n} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{27}{n} + \frac{27}{n^2}\ i\right] \end{aligned}$$

We we now perform the sum. Remember that the $\dfrac{27}{n}$ is a constant and its sum will be the constant times the upper bound. The $\dfrac{27}{n^2}$ is a coefficient and will be "carried along" as we sum the i.

$$= \lim_{n\to\infty}\left[27 + \frac{27}{n^2}\,\frac{n^2+n}{2}\right]$$
$$= \lim_{n\to\infty}\left[27 + \frac{27}{2}\,\frac{n^2+n}{n^2}\right]$$
$$= 27 + \frac{27}{2}\,(1)$$
$$= \frac{81}{2}$$

Therefore the area is $\frac{81}{2}$.

Note that in the second to last step, the limit of $\frac{n^2+n}{n^2}$ as $n \to \infty$ is 1.

We will only ask you to perform right sums when finding areas via the limit of a Riemann sum. Left sums require us to manipulate $a + (i-1)\,\Delta x$ instead of $a + i\,\Delta x$. The latter is always easier.

There are many steps in this type of problem and some can be done at various times in the process, but here are some general guidelines:

- Find Δx first.
- Begin with $A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$.
- Replace x_i with $a + i\,\Delta x$.
- Remember that the a is simply where your area begins.
- Find the value of f at $a + i\,\Delta x$.
- Start simplifying.
- At some point you need to replace the Δx with its equivalent expression with the n in the denominator.
- Do the sums. Remember the difference between summing something line $\frac{9}{n}$ as opposed to $\frac{9}{n}\,i$.
- Find the limits. Remember you may have to switch denominators in at least one term to make the limits easier to see.

Example 2

Find the exact area of the region bounded by $y = x^2 + 2$, $x = 2$, $x = 5$ and the x-axis.

$$\Delta x = \frac{5-2}{n} = \frac{3}{n}$$
$$c_i = x_i = 2 + i\,\Delta x$$

$$
\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(2 + i\ \Delta x)\ \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} [(2 + i\ \Delta x)^2 + 2]\ \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} [4 + 4i\ \Delta x + i^2(\Delta x)^2 + 2]\ \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{12}{n} + \frac{36}{n^2}\ i + \frac{27}{n^3}\ i^2 + \frac{6}{n}\right] \\
&= \lim_{n\to\infty} \left[12 + \frac{36}{n^2}\ \frac{n^2+n}{2} + \frac{27}{n^3}\ \frac{2n^3+3n^2+n}{6} + 6\right] \\
&= \lim_{n\to\infty} \left[12 + \frac{36}{2}\ \frac{n^2+n}{n^2} + \frac{27}{6}\ \frac{2n^3+3n^2+n}{n^3} + 6\right] \\
&= 12 + 18 + \frac{54}{6} + 6 \\
&= 45
\end{aligned}
$$

8.5 Definition of the Definite Integral

8.5.1 Introduction

We've finally reached the point where we can define the definite integral. Once we know this definition, we will be on the brink of our ultimate goal in this chapter… the Fundamental Theorems of Calculus. In the past several sections we have been doing a lot of work with area under a curve. Keep in mind that this is just one of the many applications of the definite integral and we are using area because it is the easiest way to get to the definition. Once we have the fundamental theorems, we will be able to apply the definite integral to a wide variety of situations.

8.5.2 Generalizing the limit of a Riemann sum

Let's start with the limit we've been using to calculate exact area under curves.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\ \Delta x$$

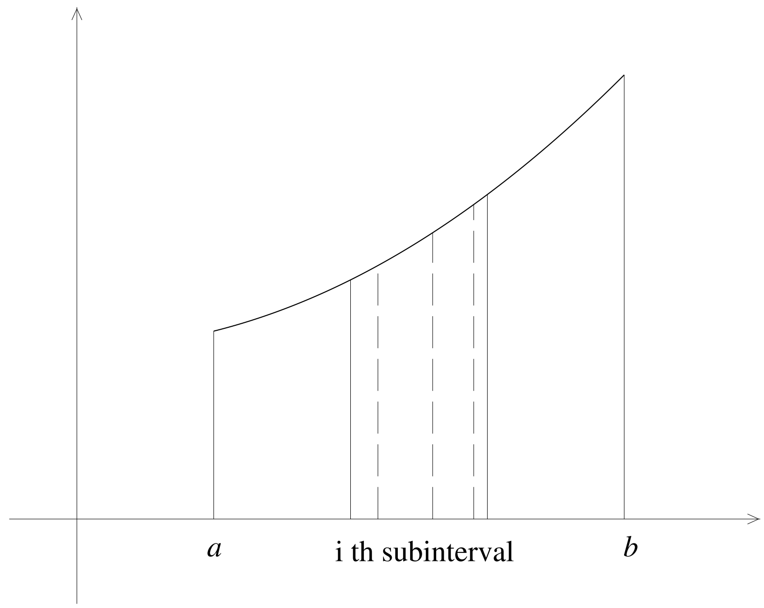
where

$$\Delta x = \frac{b-a}{n}$$

and

$$c_i = a + i\ \Delta x \text{ (right sum)}$$
or
$$c_i = a + (i-1)\ \Delta x \text{ (left sum)}$$

We already realize that, in the end, if we are finding the exact area, it does not matter whether we are doing a left sum, right sum, or midpoint sum. The exact area will be the exact area! In fact, we can use any value of x on the interval $[x_{i-1}, x_i]$ to generate the height of the rectangle. Consider the diagram below on which only the ith subinterval has been shown.



Any of the function values represented by the dashed lines could be used as the height of the rectangle. Once we let the number of rectangles go to infinity, they all become infinitely narrow and the length of the dashed lines all approach the same value.

To denote that we can actually use any value of x in $[x_{i-1}, x_i]$ to generate the heights of our rectangles, we replace the c_i (which always denotes a left or right sum) with ξ_i. This allows us to write a more general expression for the exact area under a curve.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\ \Delta x$$

This does not make the expression easier to use…it just makes it a more generalized expression.

Now, let's talk about the width of each subinterval. We've always used subintervals of equal width, namely Δx. This was just to make the summation easier. In fact, each subinterval can have its own width! Picture an interval partitioned into subintervals of varying width. The more rectangles we "pack in", the narrower they all must get. The all must shrink to arbitrarily narrow widths. To denote that each subinterval can have its own width, we replace Δx (which always denotes subintervals of equal width) with $\Delta_i x$.

In order for the limit to work with these new subintervals, we don't let the number of subinterval go to infinity as we usually do. We need a better way to get an infinite number of rectangles. Now, since all the subintervals have different width, there must be a widest one. The width of the widest subinterval is called the "norm" and is denoted $||\Delta||$. To get an infinite number of subintervals, we let the norm go to zero…we let the width of the largest subinterval shrink to zero. We can now rewrite the expression for the exact area in its final, most generalized form.

$$A = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(\xi_i)\ \Delta_i x$$

So now we have the most general form for the limit of a Riemann sum to find the exact area…but it is not very useful to us in terms of doing problems. It's just important to know that (1) any x value in the ith subinterval can be used to generate heights and (2) the subinterval really do not have be of equal width.

In order to actually find exact areas using the limit of a Riemann sum, we will always revert to

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

because it's the easiest!

8.5.3 Defining the definite integral

Although $A = \lim\limits_{||\Delta|| \to 0} \sum\limits_{i=1}^{n} f(\xi_i)\ \Delta_i x$ is a wonderfully explicit way to express the exact area under a curve, it is somewhat cumbersome. To get around the cumbersome nature of the expression, mathematicians did what they always do…create a new notation that is more compact.

The *definite integral* is then defined as:

$$\int_a^b f(x)\ dx = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(\xi_i)\ \Delta_i x$$

Up until now, we have used this definite integral strictly as a means to calculate exact area. The definite integral, however, is a mathematical object in and of itself and has many, many other applications. To find the value of a definite integral (whatever it may represent) we must still revert to our limit of a Riemann sum…until we get our Fundamental Theorems of Calculus…then, we will evaluate definite integrals much differently. (As a matter of fact, most of the definite integrals we will deal with would be tremendously difficult if we had to use the limit of a Riemann sum!)

So, for now, we will continue to use limits of Riemann sums to evaluate definite integrals. Remember, right sums are always easier!

Example 1

Evaluate $\displaystyle\int_1^3 x^2\ dx$.

This definite integral could represent many, many different things…or it could just be a problem to be evaluated…like a multiplication problem.

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}$$
$$x_i = 1 + i\ \Delta x$$

$$\begin{aligned}
\int_1^3 x^2\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(1 + i\ \Delta x)\ \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + 2i\ \Delta x + i^2\ (\Delta x)^2\right]\ \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\Delta x + 2i\ (\Delta x)^2 + i^2\ (\Delta x)^3\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2}{n} + \frac{8}{n^2}\ i + \frac{8}{n^3}\ i^2\right] \\
&= \lim_{n\to\infty} \left[2 + \frac{8}{n^2}\ \frac{n^2+n}{2} + \frac{8}{n^3}\ \frac{2n^3+3n^2+n}{6}\right] \\
&= \lim_{n\to\infty} \left[2 + \frac{8}{2}\ \frac{n^2+n}{n^2} + \frac{8}{6}\ \frac{2n^3+3n^2+n}{n^3}\right] \\
&= 2 + 4(1) + \frac{8}{6}(2) \\
&= \frac{26}{3}
\end{aligned}$$

$$\therefore \int_1^3 x^2\ dx = \frac{26}{3}$$

Example 2

Evaluate $\int_5^9 (3x+2)\ dx$.

$$\Delta x = \frac{9-5}{n} = \frac{4}{n}$$
$$x_i = 5 + i\ \Delta x$$

$$\begin{aligned}
\int_5^9 (3x+2)\ dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\ \Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(5+i\ \Delta x)\ \Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} [3(5+i\ \Delta x)+2]\ \ \Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} [15+3i\ \Delta x+2]\ \ \Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} [17+3i\ \Delta x]\ \ \Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left[17\ \Delta x+3i\ (\Delta x)^2\right] \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left[17\ \frac{4}{n}+3i\ \frac{16}{n^2}\right] \\
&= \lim_{n\to\infty}\left[68+\frac{48}{n^2}\ \frac{n^2+n}{2}\right] \\
&= \lim_{n\to\infty}\left[68+\frac{48}{2}\ \frac{n^2+n}{n^2}\right] \\
&= 68+24 \\
&= 92
\end{aligned}$$

$$\therefore \int_5^9 (3x+2)\ dx = 92$$

8.5.4 Properties of the definite integral

Like any mathematical object, the definite integral has certain properties. We present them here without proof.

$$\int_a^a f(x)\ dx = 0$$

Think in terms of area…there is no area under f from a to a!

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

This property will be very useful to us as we work more with evaluating definite integrals. There is a direction to integrating! We use this property to switch upper and lower bounds so that the lower bound is in fact smaller than the upper bound. We can also use it to remove a negative from the front of a definite integral.

$$\int_a^b k\ f(x)\ dx = \text{k} \int_a^b f(x)\ dx$$

Integrating a constant times a function is the same as integrating the function and them multiplying the result by k. This property also held for derivatives. Constants can be "brought out in front" of an integral.

$$\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

The integral of a sum is the same as the sum of the integrals.

$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$, regardless of the order of a, b and c.

This property will be extremely useful to us once we begin doing more involved area problems.

An example of this property would look like this: $\int_2^7 f(x)\ dx = \int_2^4 f(x)\ dx + \int_4^7 f(x)\ dx$.

$\int_a^b k\ dx = k(b-a)$, where k is a constant.

To integrate a constant, multiply the constant by the difference in the bounds. The is one definite integral we can evaluate without resorting to a limit of a Riemann sum or the fundamental theorems.

Example 3

Given that $\int_{-1}^{2} x^2\ dx = 3$, find $\int_{-1}^{2} (8 - x^2)\ dx$.

$$\begin{aligned}\int_{-1}^{2} (8 - x^2)\ dx &= \int_{-1}^{2} 8\ dx - \int_{-1}^{2} (x^2)\ dx \\ &= 8(2 - -1) - 3 \\ &= 21\end{aligned}$$

Example 4

Given that $\int_{-1}^{2} x^2\ dx = 3$, and $\int_{-1}^{2} x\ dx = \frac{3}{2}$, find $\int_{2}^{-1} 3x(x-4)\ dx$.

$$\begin{aligned}\int_{2}^{-1} 3x(x-4)\ dx &= \int_{2}^{-1} (3x^2 - 12x)\ dx \\ &= 3\int_{2}^{-1} x^2\ dx - 12\int_{2}^{-1} x\ dx \\ &= -3\int_{-1}^{2} x^2\ dx + 12\int_{-1}^{2} x\ dx \\ &= -3(3) + 12\left(\frac{3}{2}\right) \\ &= 9\end{aligned}$$

Example 5

Given $\int_a^b g(x)\ dx = 4a + 3b$, find $\int_a^b [5g(x) - 7]\ dx$.

$$\begin{aligned}\int_a^b [5g(x) - 7]\ dx &= 5\int_a^b g(x)\ dx - \int_a^b 7\ dx \\ &= 5(4a + 3b) - 7(b - a) \\ &= 20a + 15b - 7b + 7a \\ &= 27a + 8b\end{aligned}$$

Chapter 9

The Fundamental Theorems of Calculus

9.1 The Fundamental Theorems

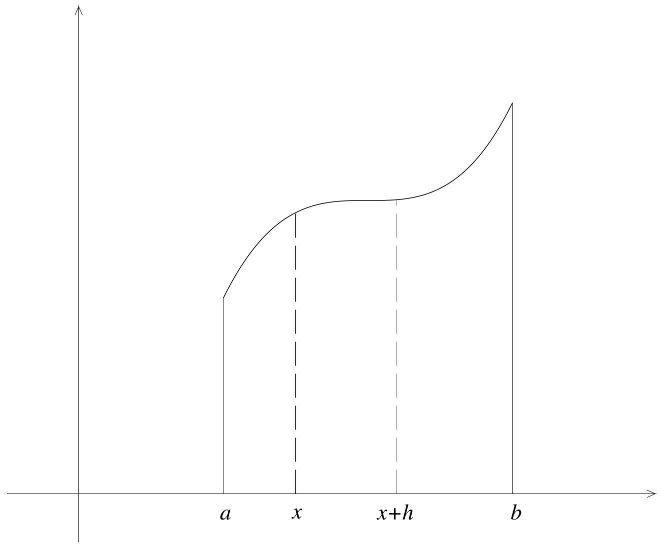
9.1.1 Introduction

All the pieces are in place. It's time to finally get to the Fundamental Theorems of Calculus. The First Fundamental Theorem tells us that the area under the graph of a function can be described by the antiderivative of the of the function which is graphed and gives us a way to easily evaluate definite integrals. The Second Fundamental Theorem is a formal statement about the relationship between integration and differentiation. Some texts will state only one fundamental theorem (the one we call the "First") and explain that the theorem we call the "Second" is simply a restatement of the first.

In the end, for us, the Fundamental Theorems are going to give us a means to evaluate definite integrals without resorting to the limit of a Riemann sum.

9.1.2 The first fundamental theorem

Consider the graph of the function f on $[a, b]$ below and the area under f from $x = a$ to $x = b$. The graph also shows a subinterval with x as a left endpoint and $x + h$ as a right endpoint.



Now,let's make up a function called A which will yield the area under f from $x = a$ to any other x value we want. We do not yet know what A looks like... we only know what it does. Now, let's play with this function a little. Consider the following:

- $A(x)$ yields the area from a to x.
- $A(b)$ will give us the area under f from a to b.
- $A(a)$ will give us the area under f from a to a, which will of course, be zero.
- $A(x + h)$ will give us the area under f from a to $x + h$.
- $A(x+h) - A(x)$ will give us the exact area of the subinterval with x as a left endpoint and $x+h$ as a right endpoint.

An approximation of the area of the subinterval with x as a left endpoint and $x+h$ as a right endpoint can be obtained using a rectangle with the left endpoint used to generate the height:

$$f(x) \cdot h$$

where h is the width of the subinterval and we use $f(x)$ for a height.

Now we can state

$$f(x) \cdot h \approx A(x + h) - A(x).$$

We can say this because $f(x) \cdot h$ is the area of the left sum rectangle in the subinterval and for a sufficiently small value of h the area of this rectangle will be almost the same as the area under f from x to $x+h$... the area of the subinterval itself.

OK, you're getting suspicious... you should be... anytime you see an expression of the form $f(x + h) - f(x)$ during a derivation you can rest assured it's not just for grins!

Let's divide both sides by h.

$$f(x) \approx \frac{A(x + h) - A(x)}{h}$$

And now the fun begins! If we take a limit as $h \to 0$ we can replace the approximate symbol with an equal sign.

$$\lim_{h\to 0} f(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$$

The left side is just $f(x)$ (it has nothing to do with h.

On the right side you should recognize the definition of derivative. The definition of derivative was given earlier as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

so the right side is just $A'(x)$!

If $f(x) = A'(x)$, then $\int f(x)\ dx = A(x) + D$! (D is our constant we add to all our antiderivatives.)

Remember that f has many antiderivatives–all different by a constant. We are going to let F be another antiderivative of f so

$$A(x) = F(x) + C$$

Now, $A(a) = F(a) + C$

But, $A(a) = 0 \therefore 0 = F(a) + C$

so

$$-F(a) = C$$

thus

$$A(x) = F(x) - F(a)$$

so

$$A(b) = F(b) - F(a)$$

We now know two things. $A(b)$ is the area under f from a to b. We also know that $F(b) - F(a)$ is the area under f from a to b.

Hang on. . . we're almost there!

In the previous section, we defined the definite integral as

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

This limit on the right is the exact area under f from $x = a$ to $x = b$. We now combine these to obtain the First Fundamental Theorem of Calculus.

The First Fundamental Theorem of Calculus

$$\int_a^b f(x)\ dx = F(b) - F(a), \text{ where } F'(x) = f(x)$$

The statement "where $F'(x) = f(x)$" simply means that F is the antiderivative of f.

We now have a method of evaluating a definite integral without having to find the limit of a Riemann sum!

The first fundamental theorem allows us to use any antiderivative of f to get F. This means we can use a constant of integration of zero…the simplest antiderivative of f. When we evaluate a *definite* integral, we do not need the C.

Note: Integrals that have no upper or lower bound are called "indefinite" integrals. We still need to add a "$+C$" to these indefinite integrals. We do not need the C when we evaluate definite integrals. In fact, if we did include a constant, when we took $F(b) - F(a)$, the constants would add to zero anyway. Take a look at this example in which we include a constant and see what happens.

$$\begin{aligned}\int_2^5 x^2\ dx &= \left(\frac{1}{3}x^3 + C\right)\Bigg|_2^5 \\ &= \left[\frac{1}{3}\ 5^3 + C\right] - \left[\frac{1}{3}\ 2^3 + C\right] \\ &= \frac{125}{3} + C - \frac{8}{3} - C \\ &= 39\end{aligned}$$

The constants will always add to zero so we do not have to use them. The previous problem is actually done like this:

$$\begin{aligned}\int_2^5 x^2\ dx &= \left(\frac{1}{3}x^3\right)\Bigg|_2^5 \\ &= \left[\frac{1}{3}\ 5^3\right] - \left[\frac{1}{3}\ 2^3\right] \\ &= \frac{125}{3} - \frac{8}{3} \\ &= 39\end{aligned}$$

Example 1

Evaluate $\displaystyle\int_{-2}^{3} x^2\ dx$.

$$\begin{aligned}\int_{-2}^{3} x^2\ dx &= \left(\frac{1}{3}x^3\right)\Bigg|_{-2}^{3} \\ &= \left[\frac{1}{3}\ 3^3\right] - \left[\frac{1}{3}\ (-2)^3\right] \\ &= \frac{27}{3} - \frac{-8}{3} \\ &= \frac{35}{3}\end{aligned}$$

Just a little easier than the limit of a Riemann sum!

Example 2

Find the area bounded by $f(x) = -x^2 + 9$ from $x = 0$ to $x = 2$.

Since this area is entirely above the x-axis, we can express the area as $\int_0^2 (-x^2 + 9)\ dx$.

Other area problems may not be so straightforward.

$$\begin{aligned}\int_0^2 (-x^2 + 9)\ dx &= \left[-\frac{1}{3}\,x^3 + 9x\right]_0^2 \\ &= \left[-\frac{1}{3}(2)^3 + 9(2)\right] - \left[-\frac{1}{3}(0)^3 + 9(0)\right] \\ &= \frac{46}{3}\end{aligned}$$

Thus the area is $\frac{46}{3}$.

(No units in the problem, so no units in the answer.)

Example 3

Evaluate $\int_0^{2\pi} \sin x\ dx$.

$$\begin{aligned}\int_0^{2\pi} \sin x\ dx &= -\cos x\Big|_0^{2\pi} \\ &= [-\cos 2\pi] - [-\cos 0] \\ &= [-1] - [-1] \\ &= 0\end{aligned}$$

Now, if we interpret the original definite integral as the area under $\sin x$ from $x = 0$ to $x = 2\pi$, our answer indicates zero area! Definite integrals are just mathematical objects that must be interpreted. In this case there are equal areas above and below the x-axis and they add to zero. What we have found is the "net area" and not "total area". If we really wanted the total area bounded by $\sin x$ and the x-axis we would have to perform a different calculation. . . more on that later.

The next step is to consider definite integrals that require substitution. You should already be comfortable with making a substitution to do an antiderivative. Doing substitution with definite integrals works the same. . . with a couple of minor but important changes.

Once we pick a u for our substitution, we need to make sure the entire problem is now in u. The original bounds are x-values and they must also be put in terms of u. *The good thing about definite integrals is that once you get the problem in terms of u you never go back to x. Never.*

Example 4

Evaluate $\int_1^3 x^2(x^3-8)^5\ dx$.

$$u = x^3 - 8 \longrightarrow du = 3x^2\ dx \longrightarrow \frac{1}{3}\ du = x^2\ dx$$
$$x = 1 \longrightarrow u = -7$$
$$x = 3 \longrightarrow u = 19$$

$$\begin{aligned}\int_1^3 x^2(x^3-8)^5\ dx &= \frac{1}{3}\int_{-7}^{19} u^5\ du \\ &= \frac{1}{18}\ u^6\Big|_{-7}^{19} \\ &= \left[\frac{1}{18}\ 19^6\right] - \left[\frac{1}{18}\ (-7)^6\right] \\ &= 2,607,124\end{aligned}$$

WARNING: One of the most common errors students make is failing to change the bounds after they pick a u. If you use substitution, you *must* change the bounds.

9.1.3 The second fundamental theorem

We now return to our discussion of the fundamental theorems. Consider the following integral.

$$\int_2^x t^2\ dt$$

The first thing we notice is that the integral is in terms of t and the upper bound is a variable. We can still use our first fundamental theorem to evaluate it.

$$\int_2^x t^2\ dt = \frac{1}{3}\ t^3\Big|_2^x = \frac{1}{3}x^3 - \frac{8}{3}$$

Now consider the following:

$$\frac{d}{dx}\int_2^x t^2\ dt.$$

We are now being asked to take the derivative of the integral.

$$\frac{d}{dx}\int_{2}^{x} t^2\ dt = \frac{d}{dx}\left[\frac{1}{3}x^3 - \frac{8}{3}\right] = x^2$$

Notice that all that really happened is that the original function t^2 changed to x^2. This is a simple application of the Second Fundamental Theorem of Calculus which explicitly states the relationship between the derivative and the integral.

The Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{a}^{x} f(t)\ dt = f(x) \text{ where } a \text{ is a constant.}$$

To apply the theorem, we do not actually have to find an antiderivative–we just change the function!

Example 5

Find $\frac{d}{dx}\int_{5}^{x} e^{t^3}\ dt$.

$$\frac{d}{dx}\int_{5}^{x} e^{t^3}\ dt = e^{x^3}$$

What would happen if the lower bound was also a function? What would happen if one or both of the bounds was not simply a single variable? Consider the following problem.

$$\frac{d}{dx}\int_{4x}^{x^3} e^{t^3}\ dt$$

A simple application of the second fundamental theorem will not work because of the nature of the bounds.

Let's do the problem in general and see what happens. We will replace the upper and lower bounds with general functions. First we will integrate and then we will differentiate. We will need to apply the chain rule! Also, we can denote the antiderivative of $f(t)$ as $F(t)$ and so $F'(t) = f(t)$.

$$\begin{aligned}\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\ dt &= \frac{d}{dx}\left[F(t)\Big|_{g(x)}^{h(x)}\right] \\ &= \frac{d}{dx}\left[F(h(x)) - F(g(x))\right] \\ &= F'(h(x))\ h'(x) - F'(g(x)\ g'(x) \\ &= f(h(x))\ h'(x) - f(g(x))\ g'(x)\end{aligned}$$

This gives us a more powerful version of our second fundamental theorem.

The Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\ dt = f(h(x))\ h'(x) - f(g(x))\ g'(x)$$

To use the theorem, we find the value of f at the upper bound, multiply by the derivative of the upper bound and subtract the valueof f at the lower bound times the derivative of the lower bound.

Example 6

Find $\frac{d}{dx}\int_{4x}^{x^3} \sin t^2\ dt$.

$$\begin{aligned}\frac{d}{dx}\int_{4x}^{x^3} \sin t^2\ dt &= \left(\sin x^6\right)\left(3x^2\right) - \left(\sin 16x^2\right)(4)\\ &= 3x^2 \sin x^6 - 4\sin 16x^2\end{aligned}$$

9.2 The Trapezoid Rule

9.2.1 Introduction

Interestingly, we now return to another method of approximating the value of a definite integral. It is typical in many calculus texts that the trapezoid rule is introduced now. Previously we've used Riemann sums (rectangles) to approximate areas under curves. Although we were approximating areas, we were actually approximating the value of a definite integral. There are, in fact, many different ways to approximate the value of a definite integral. Using rectangles is one. Trapezoids are another. The is also something called Simpson's Rule which used sectors of parabolas–but we won't get into that one!

Keep in mind that these numeric techniques (rectangles, trapezoids) the we employ are often used to approximate areas but more generally they are used to approximate the value of a definite integral–whatever that integral my represent.

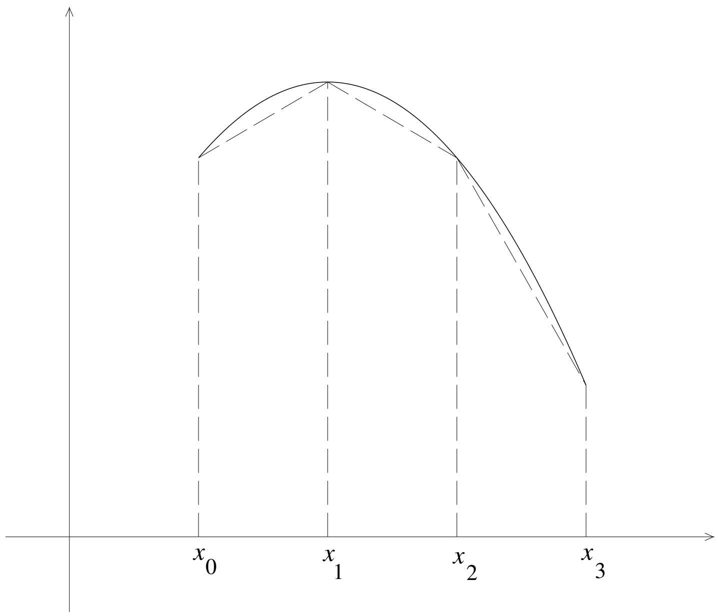
9.2.2 The trapezoid rule

The basic idea here is simple. Instead of using the left or right side of a subinterval to generate the height of a rectangle, we will connect the function values at the top of each subinterval to create trapezoids. We then add up the area of each trapezoid. To find the area of a trapezoid we use this formula:

$$A_T = \frac{1}{2}\left(b_1 + b_2\right)h.$$

It's just one-half the sum of the bases times the height. The bases are the function values and the height will always be Δx.

Take a look at the following diagram on which several subintervals have been drawn and the trapezoids created.



We can express the area under this curve by

$$\int_a^b f(x)\ dx.$$

Keep in mind that in this case the integral represents the actual area... but sometimes we just want to get an approximation for an integral and it will not matter what it represents.

So, here we go. We're just going to add up the trapezoids.

$$\int_a^b f(x)\ dx \approx \frac{1}{2}\left[f(x_0) + f(x_1)\right]\ \Delta x + \frac{1}{2}\left[f(x_1) + f(x_2)\right]\ \Delta x + \frac{1}{2}\left[f(x_2) + f(x_3)\right]\ \Delta x$$

We can factor out the one-half and the Δx and get

$$\int_a^b f(x)\ dx \approx \frac{1}{2}\ \Delta x\left[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3)\right]$$

which then can be simplified to

$$\int_a^b f(x)\ dx \approx \frac{1}{2}\ \Delta x\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

Since $\Delta x = \dfrac{b-a}{n}$ the term out in front can be changed to get:

$$\int_a^b f(x)\ dx \approx \frac{b-a}{2n}\ [f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)]$$

Notice that the first and last x-coordinates are used only once and all the x-coordinates in between are used twice.

To generalize this expression for n subintervals we get:

$$\int_a^b f(x)\ dx \approx \frac{b-a}{2n}\,[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)]$$

We can now make use of our summation notation to make the trapezoid rule even more compact.

The Trapezoid Rule

$$\int_a^b f(x)\ dx \approx \frac{b-a}{2n}\ \sum_{i=1}^{n} k_i\ f(x_i)$$

The k_i takes on the value 1 or 2 depending on what the i is.

Wow...now that we've got that let me say something about most trapezoid approximations. For the vast bulk of problems (especially on the AP test) you won't need the rule as it appears. All you need to do is calculate the area of a few trapezoids and then add them up. That simple. This is especially true in the case where we do not use subintervals of equal width! The formula we just derived is useful only on problems where the number of subintervals exceeds five or six.

Example 1

Use a trapezoid approximation to estimate the value of $\int_1^4 x^2\ dx$ with 3 subintervals of equal width.

This problem only has us adding up three trapezoids so that's just what we'll do.

$$\Delta x = \frac{4-1}{3} = 1$$

$$x_0 = a = 1$$

$$x_1 = 2$$

$$x_2 = 3$$

$$x_3 = b = 4$$

$$\int_1^4 x^2\ dx \approx \frac{1}{2}[1^2 + 2^2](1) + \frac{1}{2}[2^2 + 3^2](1) + \frac{1}{2}[3^2 + 4^2](1) = \frac{43}{2}$$

Example 2

In the table below are a selection of x-values and corresponding function values for a function f. Use the data in the table to estimate $\int_3^7 f(x)\ dx$ using a trapezoidal approximation with three subintervals indicated by the table.

x	3	4.5	6.8	7
$f(x)$	2.8	6.9	4	1.8

The first thing you should notice is that the Δx is different for each trapezoid. Once again, we will simply add up the area of the three trapezoids.

$$\int_3^7 f(x)\ dx \approx \frac{1}{2}[2.8+6.9](1.5) + \frac{1}{2}[6.9+4](2.3) + \frac{1}{2}[4+1.8](.2) = 20.390.$$

Example 3

Use a trapezoid approximation with 5 subdivisions of equal width to estimate $\int_1^4 x^2\ dx$.

When using five subdivisions, it is generally easier to arrange the data in table form.

First we find Δx.

$$\Delta x = \frac{4-1}{5} = .600$$

Our table will have columns for i, x_i, $f(x_i)$, k_i, and $k_i f(x_i)$.

i	x_i	$f(x_i)$	k_i	$k_i f(x_i)$
0	1	1	1	1
1	1.600	2.560	2	5.120
2	2.200	4.840	2	9.680
3	2.800	7.840	2	15.680
4	3.400	11.560	2	23.120
5	4	16	1	16

We need to sum the numbers in the last column. This sum is 70.600.

Now, since $\int_a^b f(x)\ dx \approx \frac{b-a}{2n} \sum_{i=1}^{n} k_i\ f(x_i)$ we can say

$$\int_1^4 x^2\ dx \approx \frac{3}{10}(70.600) = 21.180$$

Note: the $\frac{3}{10}$ came from $\frac{b-a}{2n}$.

9.2.3 Calculators and the trapezoid rule

Numeric integration (rectangles, trapezoids, etc.) lends itself to extensive use of computers and calculators. In the last example, the only column filled in without the use of the calculator was the i column. Since we used five subintervals, the i column started with zero and ended with five. Lists, storage of lists, operations on lists and summations of lists were all used to complete the table. By using the table and the list capabilities we also avoided errors caused by *premature rounding*. Premature rounding is the cause of many, many lost points on the AP test. A rounding is usually caused by a student writing down a three-decimal place result (instead of storing it and using it) and using it in subsequent calculations. The error is caused because the student only uses three decimals but the calculator–using the stored value–is using something like eighteen digits in subsequent calculations.

Again, the first column was done by hand. Here's how the other columns were done:

- First, put the function in y1. y1(x)=x∧2
- The x_1 column was created by entering seq(x,x,1,4,.6)→a.
  - This creates a sequence numbers, x's, with respect to x starting with 1, ending at 4 and increasing in increments of .6.
  - Make sure you put the →a at the end before you hit the Enter key. This will automatically store the list.
- The $f(x_i)$ column was generated by entering y(a)→b.
  - This creates a list of our function values–the heights of our trapezoids and stores the list in b.
- The k_i column was created by entering {1,2,2,2,2,1}→k
  - This creates a list (stored in k) that will be used to multiply the first and last function values by 1 and the other values by 2–as required by the trapezoid rule.
- The $k_i f(x_i)$ column was generated by entering k*b→c.
  - This creates a list (stored in c) of all our function values multiplied by the appropriate 1 or 2.
- We got the sum of the last column by entering sum(c)
- Finally we multiplied this last result by 3 and divided by 10.

9.3 Integration Summary

9.3.1 The basics

$$\int u^n \; du = \frac{u^{n+1}}{n+1} + C \; n \neq -1 \qquad \int u^n \; du = \ln|u| + C, n = -1$$

$$\int \sin u \; du = -\cos u + C \qquad \int \sec^2 u \; du = \tan u + C$$

$$\int \cos u \; du = \sin u + C \qquad \int \csc^2 u \; du = -\cot u + C$$

$$\int \tan u \; du = \ln|\sin u| + C \qquad \int \sec u \tan u \; du = \sec u + C$$

$$\int cotu \; du = \ln|\sin u| + C \qquad \int \csc u \cot u \; du = -\csc u + C$$

$$\int \ln|\sec u + \tan u| \; du = \sin u + C \qquad \int e^u \; du = e^u u + C$$

$$\int \csc u \; du = \ln|\csc u - \cot u| + C \qquad \int a^u \; du = \frac{a^u}{\ln a} + C$$

9.3.2 The shortcuts

These are shortcuts when the integrand has a linear argument. It is not necessary that you memorize or use them…although it will make your life much easier if you do. If you do not use these shortcuts, you will have to use substitution.

$$\int (Ax+B)^n \; dx = \frac{1}{A} \frac{1}{n+1} (Ax+B)^{n+1} + C$$

$$\int e^{Ax+B} \; dx = \frac{1}{A} e^{Ax+B} + C$$

$$\int a^{Ax+B} \; dx = \frac{1}{A} \frac{a^{Ax+B}}{A \ln a} + C$$

$$\int \frac{D}{Ax+B} \; dx = \frac{D}{A} \ln|Ax+B| + C$$

$$\int \cos(Ax+B) \; dx = \frac{1}{A} \sin(Ax+B) + C \text{ (Works similarly for all trigonometric functions.)}$$

9.3.3 Integrals yielding the inverse trigonometric functions

There are several special integral forms that we need to be able to recognize. The following are based on the derivatives of the inverse trigonometric functions. There are only three forms you need to know.

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\,\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\,\sec^{-1}\frac{u}{a} + C$$

To prove any of the above theorems, take the derivative of the right side and you will get the integrand on the left.

Example 1

Find $\int \frac{1}{\sqrt{4-9x^2}}\,dx$.

$$u = 3x \longrightarrow \frac{1}{3}\,du = dx$$
$$a = 2$$

$$\begin{aligned}\int \frac{1}{\sqrt{4-9x^2}}\,dx &= \frac{1}{3}\int \frac{1}{\sqrt{a^2-u^2}}\,du \\ &= \frac{1}{3}\,\sin^{-1}\frac{u}{a} + C \\ &= \frac{1}{3}\,\sin^{-1}\frac{3x}{2} + C\end{aligned}$$

Example 2

Find $\int \frac{1}{x^2-6x+13}\,dx$.

The inverse trigonometric functions are often the result of an integration in which there is a constant in the numerator and a polynomial in the denominator.

We will need to complete the square to get the integrand into a form we can use.

$$\int \frac{1}{x^2-6x+13}\ dx = \int \frac{1}{(x^2-6x+9)+13-9}\ dx$$
$$= \int \frac{1}{(x-3)^2+4}\ dx$$

$$u = x-3 \ \longrightarrow\ du = dx$$

$$a = 2$$

$$\int \frac{1}{(x-3)^2+4}\ dx = \int 1\ du^2 + a^2$$
$$= \frac{1}{a}\ \tan^{-1}\frac{u}{a} + C$$
$$= \frac{1}{2}\ \tan^{-1}\frac{x-3}{2} + C$$

We will finish up this summary with a couple of examples that involve definite integrals that require substitution.

Example 3

Evaluate $\int_0^2 x^2\sqrt{x^3+1}\ dx$.

$$u = x^3+1 \ \longrightarrow\ \frac{1}{3}\ du = x^2\ dx$$

$$x = 0 \ \longrightarrow\ u = 1$$

$$x = 2 \ \longrightarrow\ u = 9$$

$$\int_0^2 x^2\sqrt{x^3+1}\ dx = \frac{1}{3}\ \int_1^9 u^{1/2}\ du$$
$$= \frac{1}{3}\ \frac{2}{3}\ u^{3/2}\Big|_1^9$$
$$= \left[\frac{2}{9}\ 9^{3/2}\right] - \left[\frac{2}{9}\ 1^{3/2}\right]$$
$$= \frac{54}{9} - \frac{2}{9}$$
$$= \frac{52}{9}$$

Example 4

Evaluate $\int_0^5 x\sqrt{x+1}\ dx$.

$$u = x + 1 \longrightarrow du = dx$$

$$\text{Since } u = x + 1 \longrightarrow x = u - 1$$

$$x = 0 \longrightarrow u = 1$$

$$x = 3 \longrightarrow u = 4$$

$$\begin{aligned}
\int_0^5 x\sqrt{x+1}\ dx &= \int_1^4 u^{1/2}(u-1)\ du \\
&= \int_1^4 (u^{3/2} - u^{1/2})\ du \\
&= \left[\frac{2}{5}\ u^{5/2} - \frac{2}{3}\ u^{3/2}\right]_1^4 \\
&= \left[\frac{64}{5} - \frac{16}{3}\right] - \left[\frac{2}{5} - \frac{2}{3}\right] \\
&= \frac{116}{15}
\end{aligned}$$

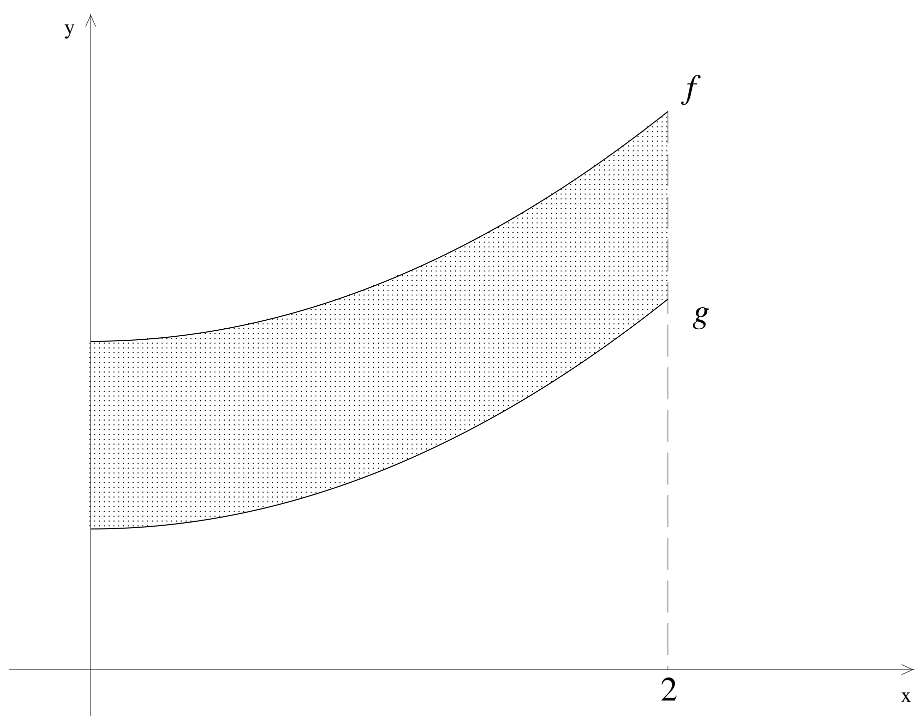
Chapter 10

Areas and Volumes

10.1 Areas Between Curves

10.1.1 Introduction

We now move on to a wider variety of problems. Up to this point we have worked with the simplest of area problems–the area under a curve when the area lies entirely above the x-axis. In order to move forward we need to combine our knowledge of the Fundamental Theorems of Calculus wit the properties of the definite integral. Consider the following graph of two functions f and g.



Suppose we wanted to find the area of the shaded region, the area between the two curves on the interval $[0, 2]$. We already know that the definite integral

$$\int_0^2 f(x)\ dx$$

will give us the area under f and the definite integral

$$\int_0^2 g(x)\ dx$$

will give us the area under g. Clearly, subtracting the area under g from the area under f will give us the area between the curves. The area of the shaded region can then be represented by the definite integral

$$\int_0^2 [f(x) - g(x)]\ dx.$$

This definite integral will be our basic tool to find areas between curves.

Although the problems will vary widely, we can list several basic problem types:

- Finding the area under one curve in which the area lies entirely above the x-axis.
- Finding the area under one curve in which the area lies entirely or partially below the x-axis.
- Finding the area between two curves when the bounds are all given.
- Finding the area between two curves when the bounds are not all given.

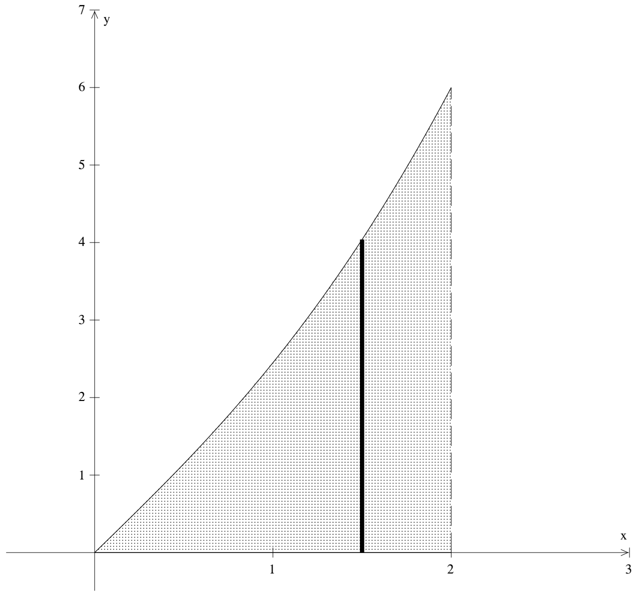
When we speak of the area "under" a curve, we actually mean the area between the function and the x-axis, or $f(x) = 0$. So, in a sense, we've been finding areas between curves all along. You will not see the term "under" very much any more. Instead, the area we are asked to find will be specified in terms of the bounds. A problem that previously stated "Find the area under $f(x) = x^2$ from $x = 2$ to $x = 5$" will now be stated "Find the area of the region bounded by $f(x) = x^2$, $x = 2$, $x = 5$ and the x-axis". If we consider all area problems as area between two curves, we can simplify our thinking by consistently saying to ourselves that the area between two curves will be the integration of the "top curve minus the bottom curve". If the problem asks for the area under a curve, we think of the bottom curve being $f(x) = 0$.

As a general guideline, it is always a good idea to work from a diagram. The diagram should include the ith rectangle. The height of this ith rectangle is the key to the problem–and all area and volume problems. Get used to drawing appropriately labeled diagrams! If you have an appropriately labeled diagram, any information you need can be obtained by simply looking at your diagram.

10.1.2 The simplest case–one curve with the area entirely above the axis

Let's find the area of the region in the first quadrant bounded by $y = x\sqrt{x^2 + 5}$, the x-axis and $x = 2$.

We begin with a sketch which includes the ith rectangle. The ith rectangle is represented by the thick vertical line inside the region.



The curves passes through the origin so the lower bound is zero. We would show this by stating that

$$x\sqrt{x^2+5} = 0 \longrightarrow x = 0$$

The top curve is the function and the bottom curve is $y = 0$ so the integral we need is

$$A = \int_0^2 \left(x\sqrt{x^2+5} - 0\right)\, dx$$

or simply

$$A = \int_0^2 x\sqrt{x^2+5}\, dx$$

This is critical: Keep in mind that all we are doing is finding the area of one rectangle and then adding up an infinite number of them–that's what integration does! In the integral we just set up, the $x\sqrt{x^2+5}$ is the height of the rectangle and dx is the width. The area of one rectangle is the height of the ith rectangle times its width dx. That's why you should always put the ith rectangle on your sketch. If you know its height, the problem is all but solved!

Back to our problem. . .

We will need substitution to evaluate this integral.

$$u = x^2 + 5 \longrightarrow \frac{1}{2}\, du = dx$$

$$x = 0 \longrightarrow u = 5$$

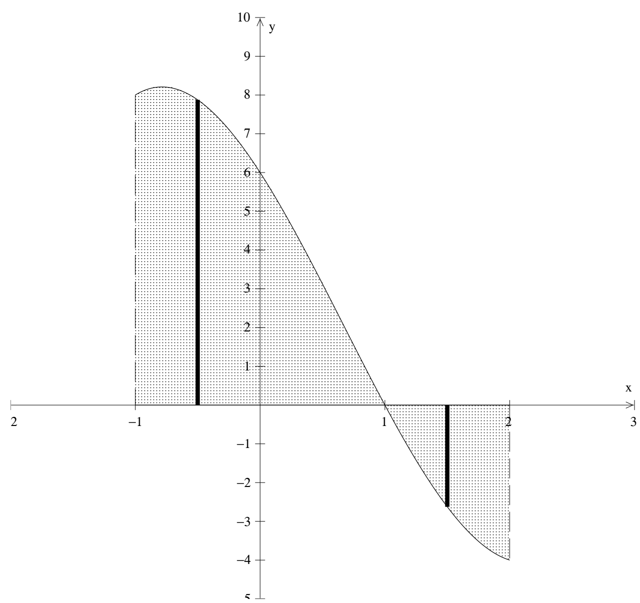
$$x = 2 \longrightarrow u = 9$$

$$\begin{aligned}\int_0^2 x\sqrt{x^2+5}\ dx &= \frac{1}{2}\int_5^9 u^{1/2}\ du\\ &= \frac{1}{2}\ \frac{2}{3}u^{3/2}\Big|_5^9\\ &= \frac{1}{3}\ 9^{3/2} - \frac{1}{3}\ 5^{3/2}\\ &= \frac{27-5\sqrt{5}}{3}\end{aligned}$$

Therefore the area is $\dfrac{27-5\sqrt{5}}{3}$.

10.1.3 Area below the axis

Find the area of the region bounded by $y = x^3 - 2x^2 - 5x + 6$, the x-axis, $x = 1$ and $x = 2$. Here is the sketch of the region.



This problem bring up another very important facet of area problems. We must know the zeros! Sometimes the zeros are the bounds and sometimes they are not! In this case they are not.

<u>Zeros</u>

$$x^3 - 2x^2 - 5x + 6 = 0 \longrightarrow x = -2 \text{ or } x = 1 \text{ or } x = 3$$

The only zero that concerns us is $x = 1$. The other zeros are outside of our interval.

The region from $x = 1$ to $x = 2$ lies below the axis and thus will be negative. In order to get the total area, this region must be subtracted.

$$A = \int_{-1}^{1} (x^3 - 2x^2 - 5x + 6)\ dx - \int_1^2 (x^3 - 2x^2 - 5x + 6)\ dx$$

This integral does not require substitution so just use the power rule and be careful with your arithmetic.

$$A = \int_{-1}^{1} (x^3 - 2x^2 - 5x + 6)\ dx - \int_{1}^{2} (x^3 - 2x^2 - 5x + 6)\ dx = \frac{157}{12}$$

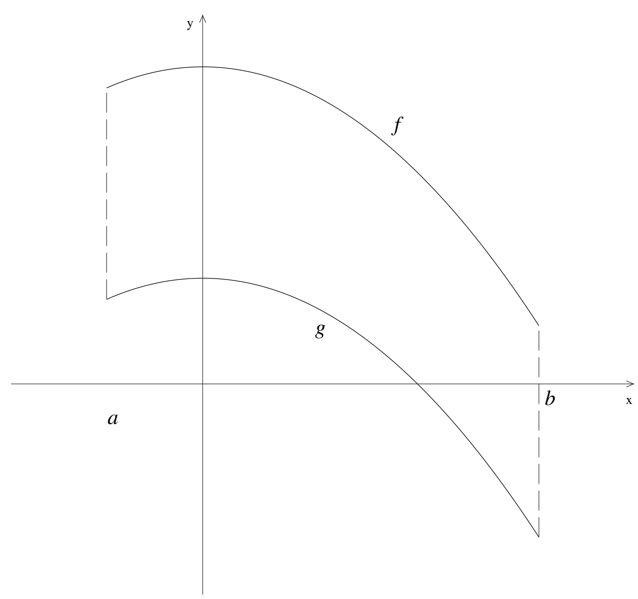
We used a sketch of the curve and the zeros to determine where the curve was above and below the x-axis. We could have determined this without a sketch. We know the zeros are $x = -2$, $x = 1$ and $x = 3$. The only zero in our interval is $x = 1$. If we picked a number between -1 and 1 and put it into the function we would get a positive value…telling us that the curve is above the axis on this interval. Putting a number between 1 and 2 into the function results in a negative value…thus this region is below the axis.

10.1.4 Area between curves

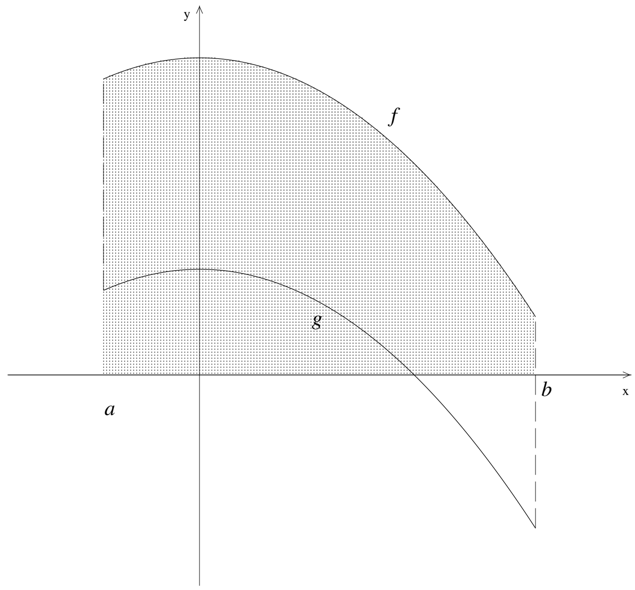
We already know that the area between two curves f and g from a to b is given by

$$\int_{a}^{b} [f(x) - g(x)]\ dx$$

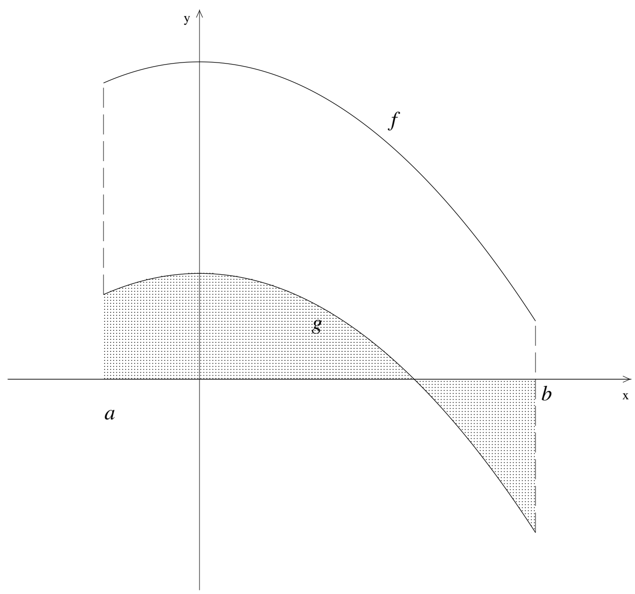
where $f(x) \geq g(x)\ \forall\ x\ \in\ (a, b)$. This integral will always yield the total area between two curves–even if all or part of the region lies below the x-axis. Consider the diagram below.



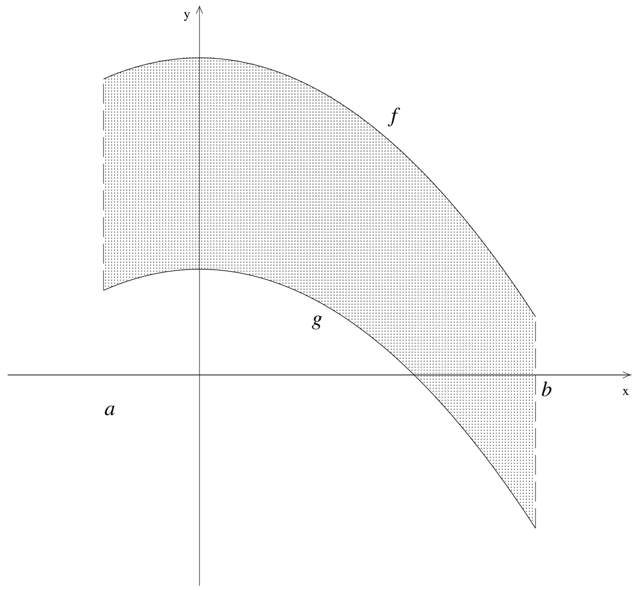
If we integrate f from a to b we would get the region as shaded below.



If we integrated g from a to b we would get the area shaded below.

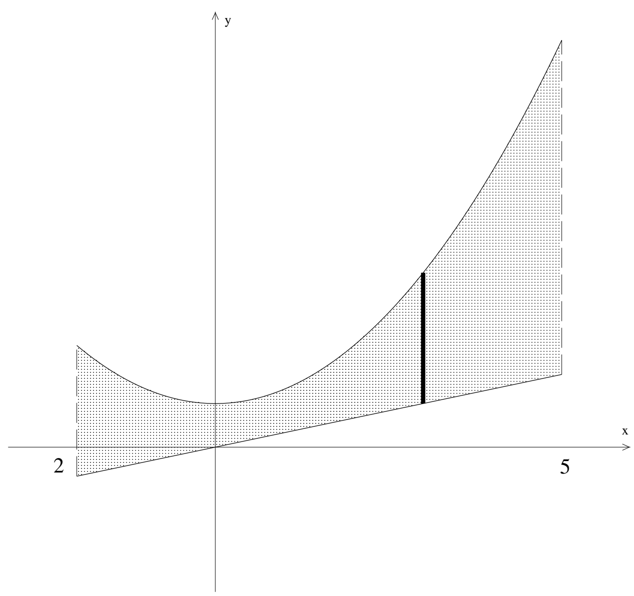


When we subtract the area "under" g from the area under f, the positive region under g on the left gets subtracted and the area below the axis gets added on. The result is the area between the two curves as shown below.



10.1.5 Area between two curves–bounds given

Find the area of the region bounded by $y = x^2 + 3$, $y = x$, $x = -2$ and $x = 5$. Below is a sketch of the area.



It is clear that the two curves do not intersect so the integral is straightforward.

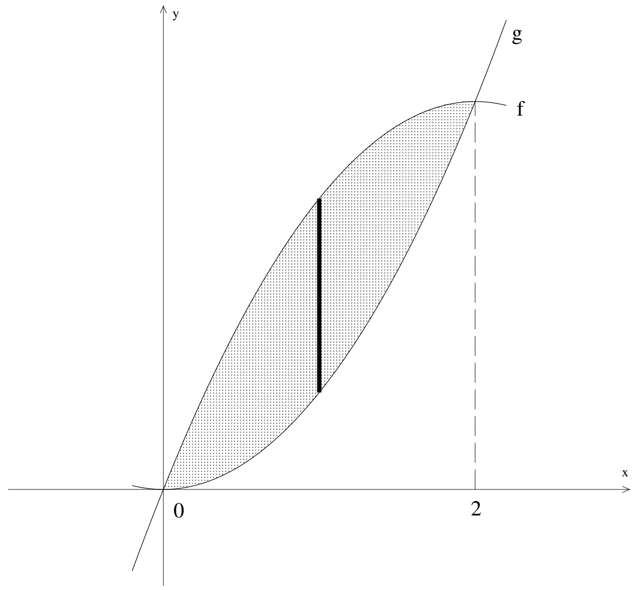
$$A = \int_{x}^{-2} [(x^2 + 3) - (x)\ d]5 = \int_{-2}^{5} (x^2 - x + 3)\ dx = \frac{329}{6}$$

Therefore the area is $\dfrac{329}{6}$.

(The actual step of integrating and evaluating are left to the reader.)

10.1.6 Area between two curves–bounds not give

Find the area of the region bounded by $f(x) = -x^2 + 4x$ and $g(x) = x^2$. Begin with a sketch as shown below.



It is clear that we need to find the intersections.

Intersections

$$-x^2 + 4x = x^2 \longrightarrow x = 0 \text{ or } x = 2.$$

From the sketch we can see that f is on top so the integral becomes

$$A = \int_0^2 [(-x^2 + 4x) - (x^2)]\, dx = \frac{8}{3}$$

Again, we leave the actual evaluation to the reader.

Worth noting...once again, $[(-x^2 + 4x) - (x^2)]$ is the height of our ith rectangle and dx is the width. Integrating adds up (accumulates) the areas of an infinite number of these rectangles.

10.1.7 Horizontal and vertical elements

The ith rectangle we draw on our diagrams is often referred to as the "element". So far we have always used vertical elements. This made the width of our rectangles dx and so, our integrals were always written in terms of x. Some problems are easier to do if we use horizontal elements instead–drawing the ith rectangle horizontally instead of vertically. This means the the width of our rectangles are all dy so our integral must be written in terms of y instead of x.

We will do the next problem first vertically, then horizontally.

Find the area of the region bounded by $y^2 = 2x - 2$ and $y = x - 5$.

Whether we do the problem vertically or horizontally, we will need to find the intersections. This is easiest to do by solving both equations for x and setting them equal.

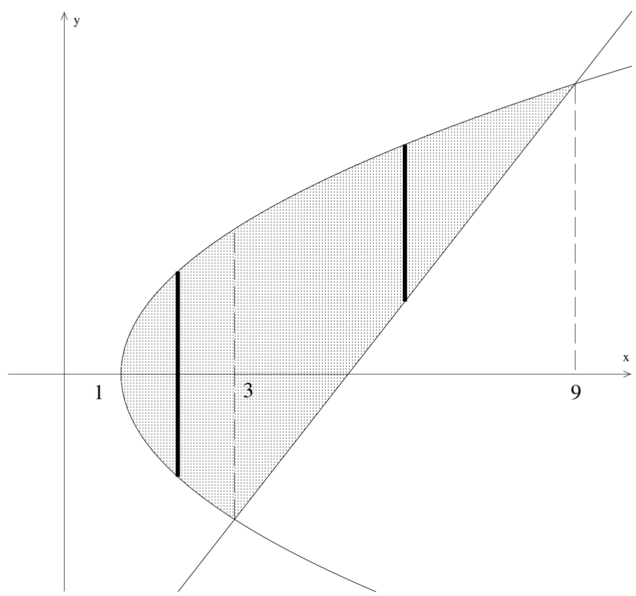
$$y = x - 5 \longrightarrow x = y + 5$$

$$y^2 = 2x - 2 \longrightarrow x = \frac{y^2 + 2}{2}$$

$$y + 5 = \frac{y^2 + 2}{2} \longrightarrow y = -2 \text{ or } y = 4$$

This gives the intersections $(3, -2)$ and $(9, 4)$.

In the sketch below you will see the area and two elements drawn in. We need two elements because for $1 \le x \le 3$, the curve on top is $\sqrt{2x-2}$ and the bottom curve is $-\sqrt{2x-2}$. However, on $3 \le x \le 9$, the top curve is $\sqrt{2x-2}$ and the bottom in now the line $x - 5$. Anytime you element changes curves as it "moves" through the region, multiple integrals will be needed. In this case we need one integral for $x = 1$ to $x = 3$ and another one for $x = 3$ to $x = 9$.



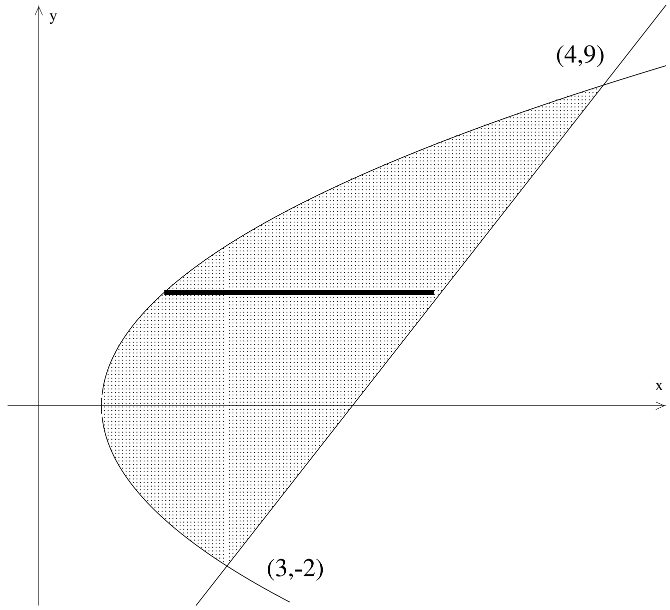
The setup to calculate the area now becomes (the steps in evaluating are now shown):

$$A = \int_1^3 \left[\left(\sqrt{2x-2}\right) - \left(-\sqrt{2x-2}\right)\right]\, dx + \int_3^9 \left[\left(\sqrt{2x-2}\right) - (x-5)\right]\, dx = 18$$

Thus the area is 18.

As you can see, the actual evaluation would be tedious at best!

We will now do the problem by putting in our ith rectangle–our element–horizontally as seen in the diagram below.



Note that no matter where we draw our element, the “top” (the curve furthest to the right) is always on the line and the “bottom” of our element is always on the parabola $y^2 = 2x - 2$. This means that only one integral will be needed. Because our element is horizontal, we must do the problem in y instead of x. Let’s solve each equation for x.

$$y^2 = 2x - 2 \longrightarrow x = \frac{y^2+2}{x}$$
$$y = x - 5 \longrightarrow x = y + 5$$

The height of our rectangle is now

$$(y+5) - \left(\frac{y^2+2}{x}\right).$$

We must also change our bounds to y. The area now extends from -2 to 4. The integral so find the area is then

$$A = \int_{-2}^{4} \left[(y+5) - \left(\frac{y^2+2}{2}\right)\right] dy.$$

This integral can be rewritten as

$$A = \int_{-2}^{4} \left[(y+5) - \left(\frac{1}{2}y^2 + 1\right)\right] dy$$

or, better yet

$$A = \int_{-2}^{4} \left(4 + y - \frac{1}{2}\, y^2\right) dy.$$

Of course, if we evaluate this integral we will still get an area of 18. This integral however, only requires our power rule and is much less complicated.

10.1.8 Summary

To find the area between two curves keep the following in mind:

- Draw a sketch.
- Draw the element–the ith rectangle.
- If you are not given the bounds you must find them–usually be intersecting the curves.
- Always determine if and where the curves intersect.
- Find the zeros of all the functions in the problem–this is especially critical if the zeros happen to lie within given bounds.
- If you set up a problem vertically and you see that the element "switches" curves, try to do the problem horizontally.

10.2 Volumes of Solids with Known Cross Sections

10.2.1 Introduction

The basic process we've been using to find areas between curves is to find the area of one rectangle, then integrating to add up an infinite number of rectangles. We now extend this idea into three dimensions to find the volumes of various solids. Volume problems basically fall into two categories: (1) volumes of solids that are sitting on the xy-plane, called solids with known cross sections and (2) volumes of solids generated when regions are rotated around a vertical or horizontal line. The section will address volumes of solids with known cross sections.

10.2.2 Volumes by "slicing"

The process of finding the volume of a solid with a known cross section is, at least theoretically, fairly simple. This idea is to slice the object into an infinite number of slices–much like we divided up areas into an infinite number of rectangles. Then we find the volume of the ith slice and integrate this to add up an infinite number of slices.

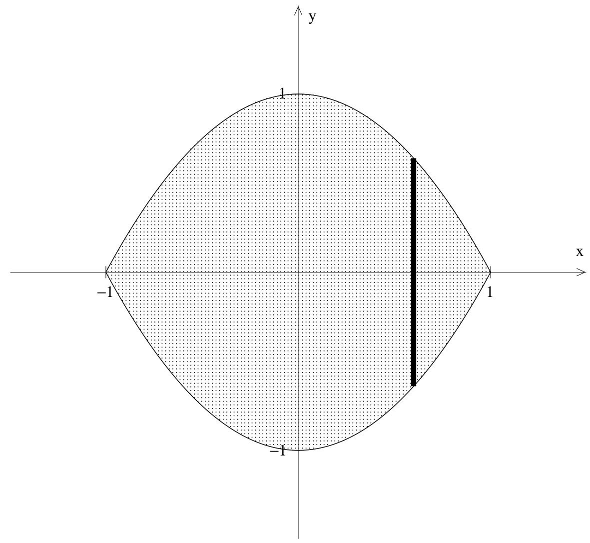
A word about our slices...they are infinitely "skinny" so that we consider them to be right solids. The volume of a right solid is

$$V = (\text{area of the base})(\text{height})$$

The height of our slice will be either dx or dy. The area of the base will be the area of the "face" of the cross section.

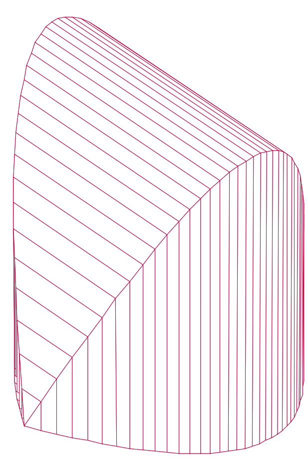
Let's try to find the volume of the solid whose base region is bounded by $y = -x^2 + 1$ and $y = x^2 - 1$ and whose cross sections taken perpendicular to the x-axis are squares.

Below is a diagram which shows a region bounded by two curves $y = -x^2 + 1$ and $y = x^2 - 1$. The slice (or element) is also shown.



The solid is sitting on the coordinate plane and sticking up at us. Imagine that we sliced the end of it off–right where the element is shown. If we put our heads down on the x-axis and looked toward the origin, we would see a square cross section.

The solid itself would look something like this:



We now find the volume of one slice. The volume is the area of the cross section multiplied by the thickness–which is dx. The area of a square is one side squared. The length of the element *is* the length of one side of the square. The length of the element is

$$(-x^2 + 1) - (x^2 - 1) = 2 - 2x^2.$$

The area of the cross section then becomes

$$(2 - 2x^2)^2.$$

The volume of our slice is now

$$(2-2x^2)^2\ dx.$$

If we integrate this from -1 to 1 (the intersections of the curves) we accumulate an infinite number of slices and we get the volume of the solid.

$$V=\int_{-1}^{1}(2-2x^2)^2\ dx=\frac{64}{15}.$$

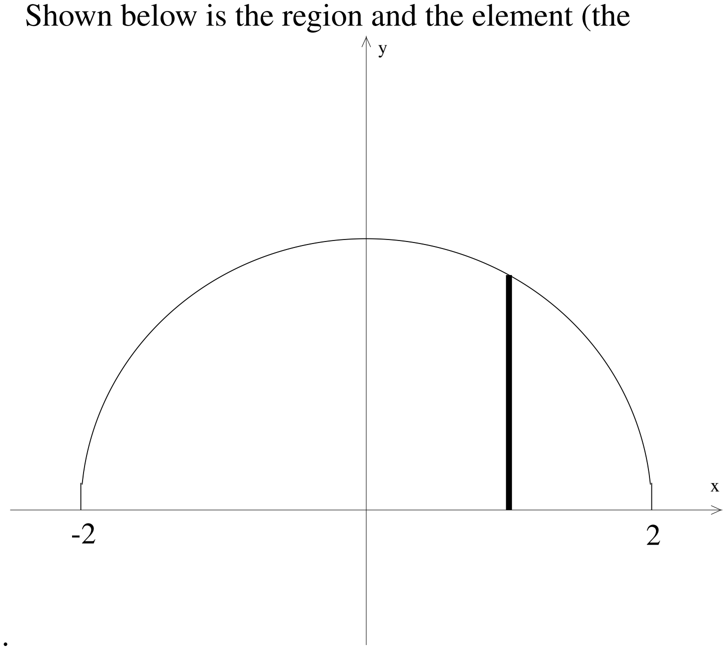
Therefore the volume of this solid is $\frac{64}{15}$.

Try to remember that all you really need to do is find the area of the cross section, multiply it by dx or dy (if doing the problem horizontally) and then integrate! The hard part, if there is one, if finding the area of the cross section. *Finding the area of the cross section depends upon finding the length of the element!* Finding the length of the element will be critical to all volume problems.

Example 1

Find the volume of the solid whose base region is bounded by $f(x)=\sqrt{4-x^2}$ and the x-axis and whose cross sections taken perpendicular to the x-axis are equilateral triangles.

Shown below is the region and the element (the



slice).

The area of the cross section of the slice is $\frac{\sqrt{3}}{4}(base)^2$–the area of an equilateral triangle. The "height" of our slice is dx and the length of the base is $\sqrt{4-x^2}$ (the height of the element) so the area of the cross section is

$$A=\frac{\sqrt{3}}{4}\left(\sqrt{4-x^2}\right)^2$$

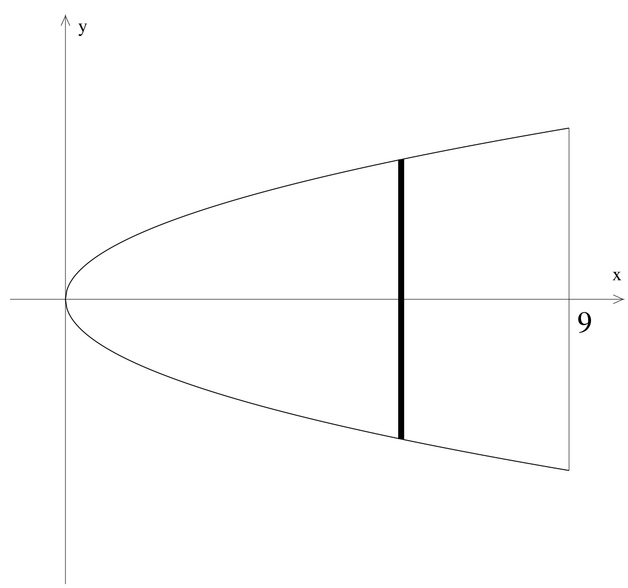
The zeros of the base region are $x = -2$ and $x = 2$ so the integral to find the volume is

$$\begin{aligned} V &= \frac{\sqrt{3}}{4} \int_{-2}^{2} \left(\sqrt{4-x^2}\right)^2 \, dx \\ &= \frac{\sqrt{3}}{4} \int_{-2}^{2} \left(4-x^2\right) \, dx \\ &= \frac{\sqrt{3}}{4} \left[4x - \frac{1}{3}\, x^3\right]_{-2}^{2} \\ &= \frac{\sqrt{3}}{4} \left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right] \\ &= \frac{8\sqrt{3}}{4} \end{aligned}$$

Therefore the volume is $\dfrac{8\sqrt{3}}{4}$.

Example 2

Find the volume of the solid whose base region is bounded by $x = y^2$ and $x = 9$ and whose cross sections taken perpendicular to the x-axis are semicircles.



In order to do this problem in terms of x we need to solve the original equation for y. This results in two curves $y = \sqrt{x}$ and $y = -\sqrt{x}$. The positive one is the upper branch and the negative one is the lower branch.

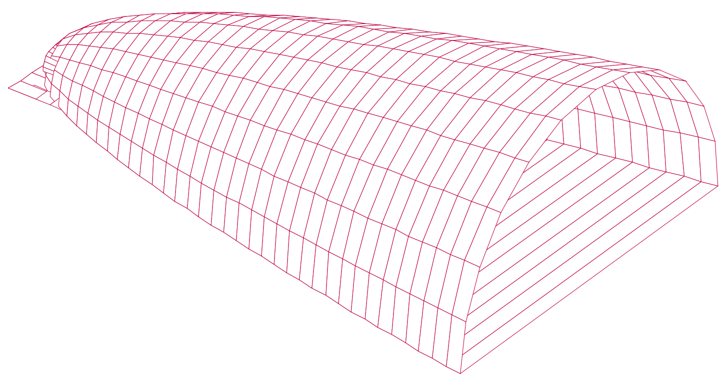
The solid is sitting on the coordinate plane and if we sliced it where the element is located, and looked toward the origin, we would see semicircles.

The area of a semicircle is just one-half the area of a circle or $\frac{1}{2}\pi r^2$ The radius of the semicircle is one-half of the length of the element–or just the distance from the x-axis to the upper branch which is $\sqrt{x}$. This is a common error when using semicircles–make sure you use the radius of the semicircle and not the diameter. The thickness of the slice is dx thus the integral is

$$V = \frac{\pi}{2} \int_0^9 \left(\sqrt{x}\right)^2 \, dx$$

Note that the $\frac{\pi}{2}$ was brought out in front of the integral because it is just a constant.

The diagram below shows the three dimensional view of this object.



To evaluate this integral:

$$\begin{aligned} V &= \frac{\pi}{2} \int_0^9 \left(\sqrt{x}\right)^2 \, dx \\ &= \frac{\pi}{2} \int_0^9 x \, dx \\ &= \frac{\pi}{2} \frac{1}{2} x^2 \Big|_0^9 \\ &= \frac{\pi}{4} x^2 \Big|_0^9 \\ &= \left(\frac{81\pi}{4}\right) - 0 \\ &= \frac{81\pi}{4} \end{aligned}$$

Therefore the volume is $\frac{81\pi}{4}$.

Example 3

Find the volume of the solid whose base region is bounded by $x = y^2$ and $x = 9$ and whose cross sections taken perpendicular to the x-axis are rectangles of height 2.

This problem uses the same region as described in Example 2. The cross sections are now rectangles so the area of the face of the cross section is length times height. The length of the rectangle is the length of the element, the height is 2 and the thickness is dx.

The length of the element is $2\sqrt{x}$ so the volume of one slice is

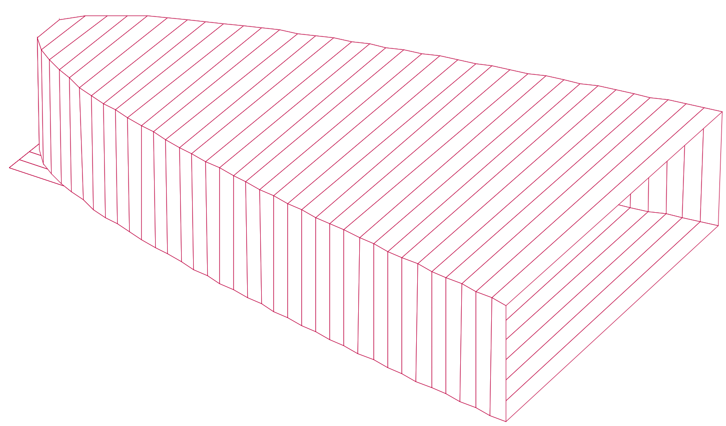
$$2(2\sqrt{x})dx$$

Integrating from 0 to 9 will add up an infinite number of our slices so the integral to calculate the volume becomes

$$V = \int_0^9 2(2\sqrt{x})\ dx = 4\int_0^9 \sqrt{x}\ dx = 72$$

Therefore the volume is 72.

Below is a three-dimensional view of the object.



10.2.3 Final note on volumes of known cross sections

Keep in mind that the definite integral allows us to add up an infinite number of slice–just like it allowed us to add up an infinite number of rectangles to find an area. To determine the expression we want to integrate, remember that the volume of a right solid is the area of the base times the height. A right solid is a solid with parallel sides–all of our slices are very flat right solids. The height of our solids is always either dx or dy and most of the time it will be dx. The area of the base is the area of the cross section in question.

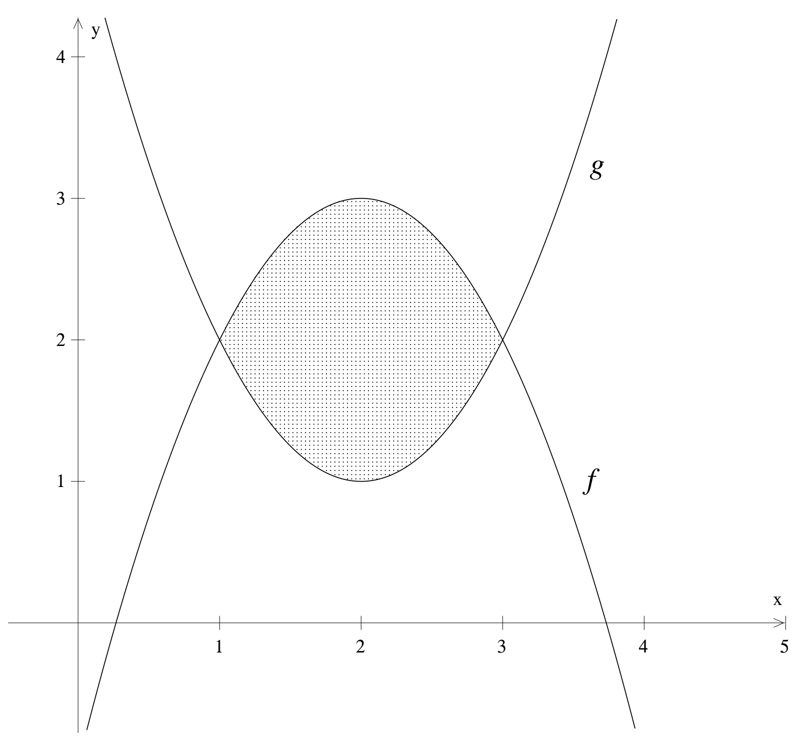
10.3 Volumes of Revolution–Disk/Washer Method

10.3.1 Introduction

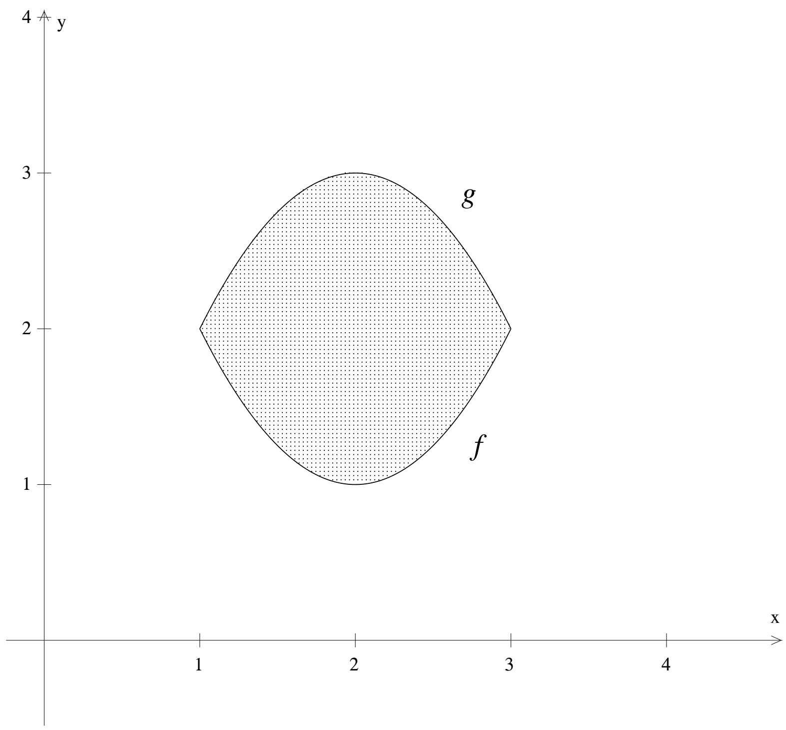
A solid of revolution is a solid generated when a particular region in the x-y plane is rotated about some vertical or horizontal line. We have two basic techniques for finding such volumes: the disk/washer method and the shell method.

10.3.2 Solids of revolution

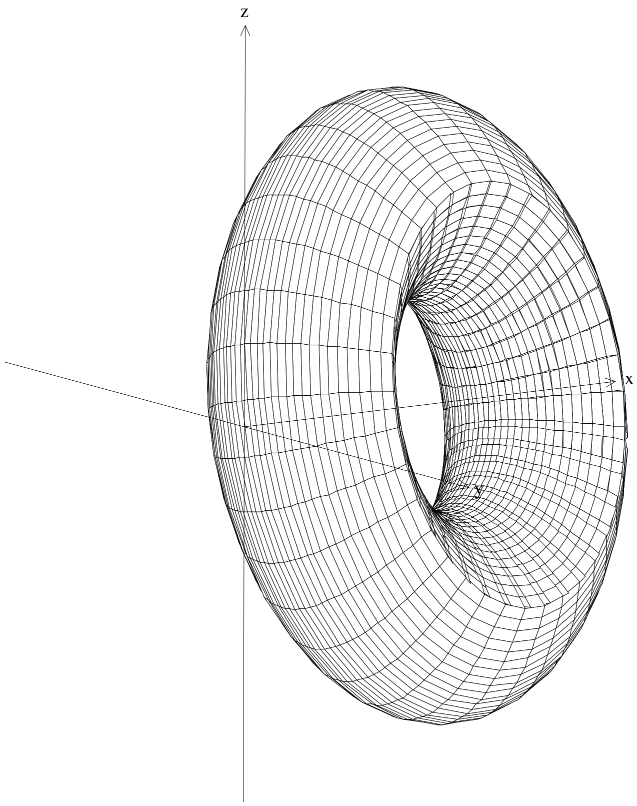
One of the biggest problems students face is visualizing solids of revolution–although it is not necessary in order to do the problem. Consider the region bounded by $f(x) = -(x-2)^2$ and $g(x) = (x-2)^2 + 1$. These curves intersect at $x = 1$ and $x = 3$.



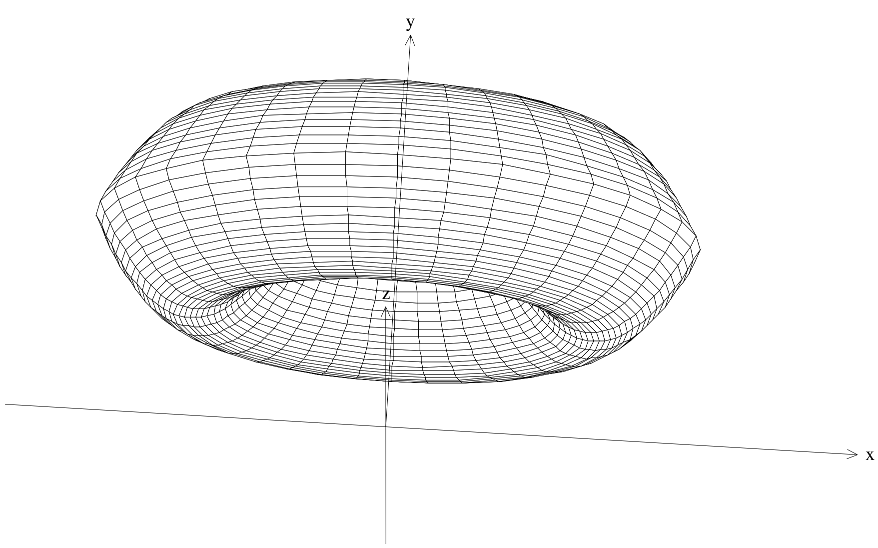
In the diagram below we have eliminated those portions of the graphs which are not in $[1, 3]$.



We will now rotate this region about the x-axis–generating a solid. Imagine a bar attaching the region to the x-axis and then rotate the region about the axis. The resulting solid is shown in the following diagram. It has been tilted so that you can actually see the nature of the solid.



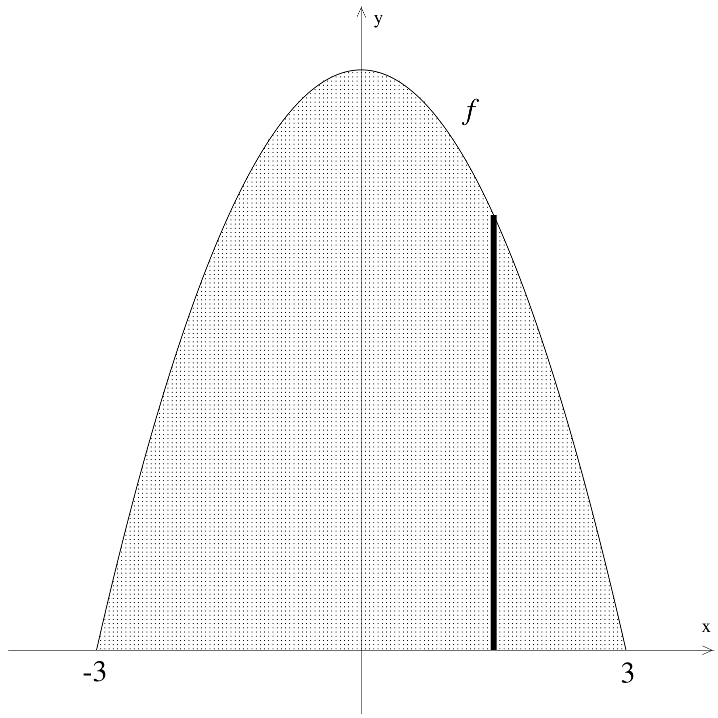
Rotating the same region about the y-axis would produce the solid below. Again, we have tilted the solid so that it is more visible.



10.3.3 Using disks to find volume

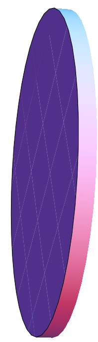
First let me say that you use a disk or washer whenever the element is perpendicular to the axis of rotation. If the element is "attached" to the axis of rotation, you get a disk. If the element is not attached to the axis of rotation you'll get a washer. If the element is parallel to the axis of rotation, you get something called shells. You have a choice between using the disk/washer method or the shell method. Once you decide to use the disk/washer method you have no choice–you either get a disk or washer. Some problems avail themselves to the disk/washer method and others will be easier using shells.

Let's begin with the parabola $f(x) = 9 = x^2$ and consider the region bounded by f and the x-axis. The dark line inside the region is the element–the ith rectangle.



Imagine now just the element rotating about the x-axis. It would sweep out a shape we call a disk. The process is the same as for areas and volumes by slicing. If we can get an expression for the volume of one disk, we can then integrate to add up an infinite number of them.

Now, a disk is just a very flat, right circular cylinder as can be seen in the diagram below.



The volume of a right cylinder is $\pi r^2 h$–the area of the base (a circle) times the height. Our disks are very short! In fact their "height" is always dx or dy.

Back to our problem. The radius of our disk is the distance from the x-axis to f, $9 - x^2$ and the height is dx.

To calculate the volume of the solid, we integrate to add up an infinite number of these disks.

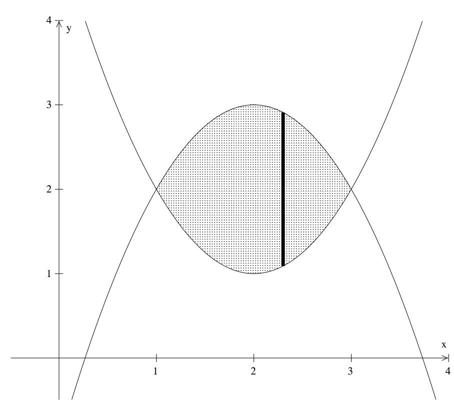
$$V = \pi \int_{-3}^{3} (9 - x^2)^2 \, dx = \frac{1296\pi}{5}$$

Therefore the volume of the solid is $\frac{1296\pi}{5}$. (The actual evaluation of the integral is left to the reader.)

When you are using disks just remember "pi radius squared thickness" and you'll be fine.

10.3.4 Using washers to find volume

Let's return to the region with which we began this section–the region bounded by $f(x) = -(x-2)^2 + 3$ and $g(x) = (x-2)^2 + 1$. We will calculate the volume of the solid with this region is revolved about the x-axis. The diagram below shows the region with the element drawn in. Remember, the intersections occurred at $x = 1$ and $x = 3$.



As this region is revolved about the x-axis, the element sweeps out a disk with a hole in it. This is called a *washer*. The volume of a washer is simply a disk with a smaller disk removed from the middle. The volume thus becomes the volume of the large disk minus the volume of the smaller disk. Remember, the volume of a disk is the same as the volume of a right cylinder: $\pi(radius)^2(height)$. If the larger disk has a radius or r_o and the smaller disk has radius r_i, the volume of the washer is

$$\pi(r_o)^2h - \pi(r_i)^2h.$$

Now, also remember that our heights are either dx or dy. In this case, since our element is vertical, the height is dx. The volume of the washer now becomes

$$\pi\left[(r_o)^2 - (r_i)^2\right]dx.$$

All we need is the actual radii and then an integration from $x = 1$ to $x = 3$ in order to add up an infinite number of washers. . . just like rectangles and disks.

In this example the outer radius of the washer is the distance from the x-axis (the axis of rotation) to the upper curve. The inner radius of the washer is the distance from the x-axis to the bottom curve. The volume of one disk now becomes

$$\pi\left[\left[-(x-2)^2+3\right]^2 - \left[(x-2)^2+1\right]^2\right]dx.$$

Finally, integrating this expression from $x = 1$ to $x = 3$ will yield the volume of the solid

$$V = \pi\int_1^3 \left[\left[-(x-2)^2+3\right]^2 - \left[(x-2)^2+1\right]^2\right]dx.$$

10.3.5 General notes

If the element is "attached" to the axis of rotation, a disk will result. If the element is not attached to the axis of rotation, we get a washer.

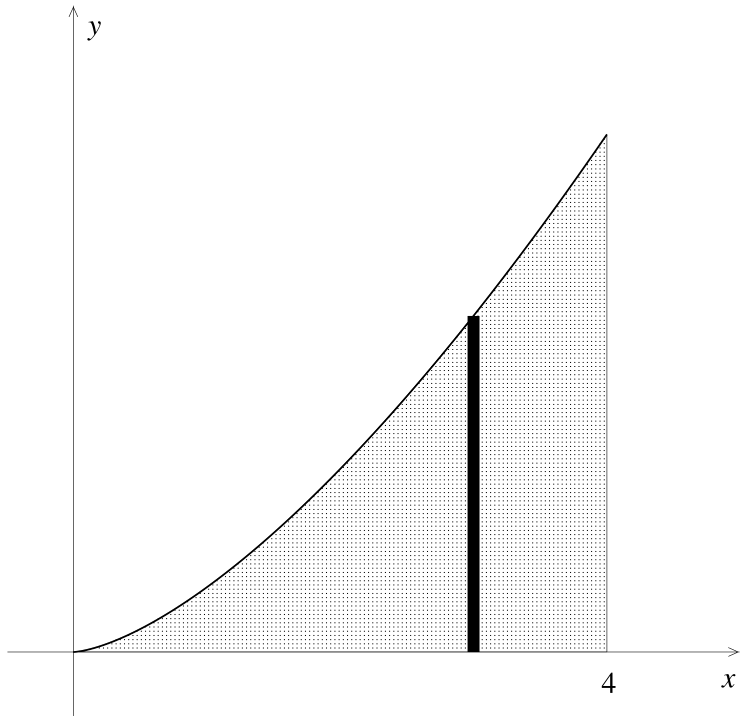
If you have a disk just remember "pi times radius squared times the thickness" where the radius is the distance from the axis of rotation to the outside of the disk. The thickness is always dx or dy.

If you have a washer remember "pi times (outer radius squared minus inner radius squared) times the thickness." The outer radius is the distance from the axis of rotation to the outside of the washer. The inner radius is the distance from the axis of rotation to the inside of the washer. Again, the thickness is either dx or dy.

Example 1

Find the volume of the solid generated when the region bounded by $y^2 = x^3$, the x-axis and $x = 4$ is rotated about the x-axis.

Below is a diagram of the region and the element.



When this region is rotated about the x-axis, the element sweeps out a disk. Since the element is vertical, we will do this problem in terms of x.

$$y^2 = x^3 \longrightarrow y = \sqrt{x^3} = x^{3/2}$$

We use only the positive root because the region lies in the first quadrant.

The radius of our disk is $x^{3/2}$.

We can now calculate the volume.

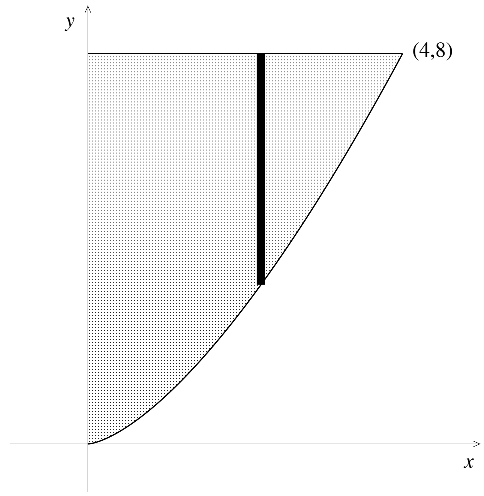
$$V = \pi \int_0^4 \left(x^{3/2}\right)^2 \, dx = 64\pi$$

Example 2

Set up but do not evaluate an integral which will yield the volume of the solid generated when the region bounded by $y^2 = x^3$, $y = 8$ and $x = 0$ is rotated about the x-axis.

These are fairly common questions on AP exams. The "set up but do not evaluate" type.

We begin by finding the intersections by solving $y^2 = x^3$ for y (and using the positive branch) setting it equal to 8. This yields $x = 4$ so our bounds of integration are 0 to 4.



Again, the element is vertical so we do the problem in x. As this region rotates about the x-axis, the element sweeps out a washer so the critical part of the integral is outer radius squared minus inner radius squared.

The outer radius is the distance from the x-axis to the line $y = 8$. The inner radius is the distance from the x-axis to the curve.

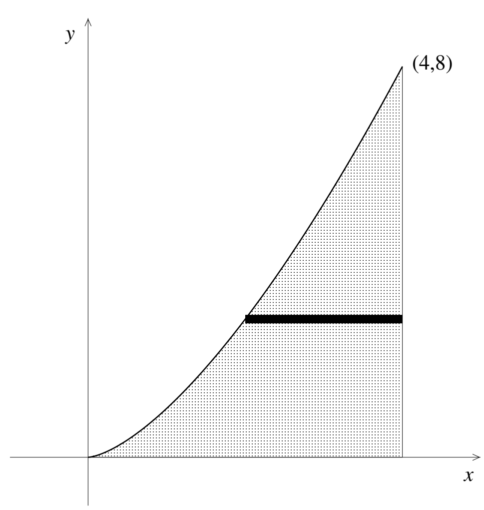
The integral to obtain the volume is therefore

$$V = \pi \int_0^4 \left[(8)^2 - (x^{3/2})^2\right] \, dx.$$

Example 3

Set up but do not evaluate an integral which will yield the volume of the solid generated when the region bounded by $y^2 = x^3$, the x-axis and $x = 4$ is rotated about the y-axis.

First of all, in order to generate a disk or washer, the element must be drawn in horizontally as shown below. Remember that for disks and washers, the element is always perpendicular to the axis of rotation. This also means that the thickness of the element is dy, not dx.



Because we are now doing the problem in y, we need to solve $y^2 = x^3$ for x to get our curve in terms of y.

$$y^2 = x^3 \longrightarrow x = y^{2/3}.$$

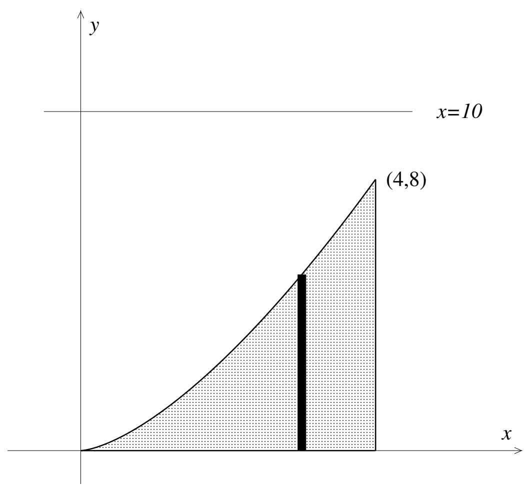
The inner radius of the washer is the distance from the y axis to the curve. This distance is $y^{2/3}$. The outer radius of the washer is the distance from the y axis to the vertical line $x = 4$, 4.

Since we are integrating in terms of y, our bounds must also be in terms of y. The region extends from 0 to 8 so those are our bounds. Putting it all together yields the integral for the volume.

$$V = \pi \int_0^8 \left[(4)^2 - (y^{2/3})^2\right] \, dy.$$

Example 4

Set up but do not evaluate an integral which will yield the volume of the solid generated when the region bounded by $y^2 = x^3$, the x-axis and $x = 4$ is rotated about the line $x = 10$.



To get a disk or washer, the element has been placed vertically in the region. The intersections are the same as previous examples. Once the element is rotated about $x = 10$, a washer is generated. The inner and outer radii must be measured from the line $x = 10$ to the inside and outside of the washer.

The inside radius will be 10 minus the distance from the x-axis to the curve, $x^{3/2}$.

The outside radius will be the distance from $x = 10$ to the outside of the washer which is the x-axis. This distance is 10.

We are integrating with respect to x so the bounds are once again 0 to 4.

The integral to calculate the volume becomes

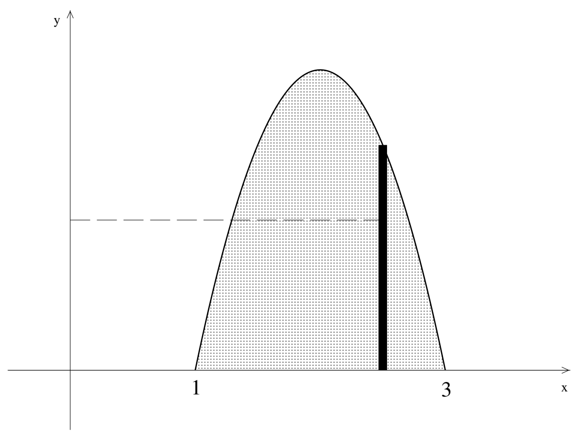
$$V = \pi \int_0^4 \left[(10)^2 - (10 - x^{3/2})^2\right] \, dx.$$

10.4 Volumes of Revolution–Shell Method

10.4.1 Introduction

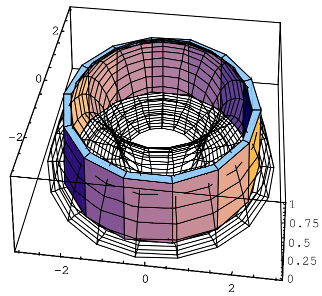
The basic difference between the disk/washer method and the shell method is in the relationship between the element and the axis of rotation and the resulting shape after rotation. For disks/washers the element is always perpendicular to the axis of rotation. For shells, the element is always placed parallel to the axis of rotation. When the element is perpendicular to the axis of rotation we get a washer or disk. When the element is parallel to the axis of rotation, we get something that looks like a piece of pipe–called a "shell".

Consider the solid generated when the region bounded by $y = -x^2 + 4x - 3$, $x = 1$, $x = 3$ and $y = 0$ is rotated about the y-axis. If we drew our element perpendicular to the y-axis, we would generate a washer. This time we will put our element parallel to the axis of rotation and generate something called a shell. Below is a diagram of the region and the element. The dashed line is the radius of the shell… more on this later.

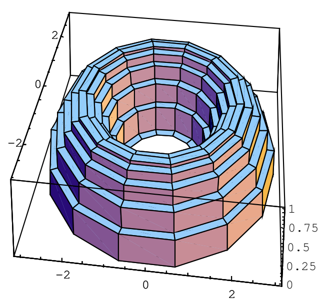


The point where the element touches the curve can be labeled $(x, -x^2 + 4x - 3)$. The dashed line is the distance from the axis of rotation to the shell which in this problem would be x. Visually, the radius acts like an arm on which the element swings around the y-axis. Notice that The thickness of the shell is dx… which is different from height of the shell. It's almost like a shell is a very tall washer!

The diagram below shows the shell. The graphic actually approximates the shell as a many-sided polygon instead of a smooth surface but you get the idea! The wire frame is just showing the "frame" of the actual solid.



The next diagram shows an approximation of the volume of the solid using ten shells.



As we did with rectangles to find area, slices and disks/washers to find volumes, we will find the volume of one shell and then integrate to find a volume.

10.4.2 Using shells to find volume

You can see that a shell looks like a piece of pipe. The pipe walls are arbitrarily narrow. Because they are so narrow we can obtain an expression for the volume by cutting the shell open lengthwise and flattening it out. We can do this without distortion because of the arbitrarily thin shell walls. If you lay it flat, we have a very thin rectangular solid...think of finding the volume of a piece of paper. The volume will be the length of the paper times the height times the thickness. The thickness will always be either dx or dy. The height is the length of the element. The length of our piece of paper is the distance around the original shell–the circumference of a circle with radius r. The radius is the distance from the axis of rotation to the element. Putting it all together, the volume of one shell will be: (2π)(radius)(height)(thickness).

We began this discussion considering the solid generated when the region bounded by $y = -x^2 + 4x - 3$, $x = 1$, $x = 3$ and $y = 0$ is rotated about the y-axis. We can now fill in the details and calculate the volume using shells.

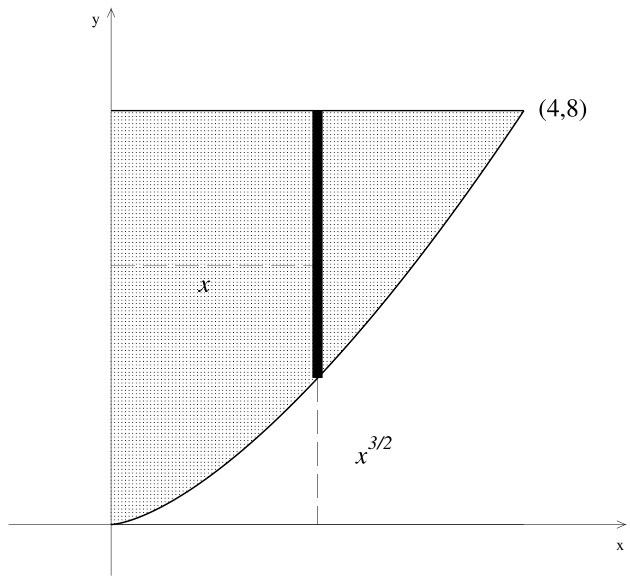
The radius is the distance from the y-axis to the shell: x. The height of the shell is the length of the element:

$-x^2+4x-3$. The thickness is dx. The volume of one shell then becomes: $2\pi(x)(-x^2+4x-3)\ dx$. Integrating this from $x=1$ to $x=3$ will accumulate an infinite number of these shells and yield the volume.

$$V = 2\pi\int_1^3 \left[(x)(-x^2+4x-3)\right]\ dx = 2\pi\int_1^3 (-x^3+4x^2-3x)\ dx$$

Example 1

Using the shell method, find the volume of the solid generated when the region bounded by $y^2=x^3$, $y=8$ and $x=0$ is rotated about the y-axis.



We must put our element in vertically so that it is parallel to the axis of rotation.

We must also have the problem in terms of x.

$$y^2 = x^3 \longrightarrow y = x^{3/2}$$

The curves $y=8$ and $y^2=x^3$ intersect at $x=4$.

The radius of the shell is just x… the distance from the axis of rotation (the y-axis) to the element.The height of the shell is the length of the element: $8-x^{3/2}$.

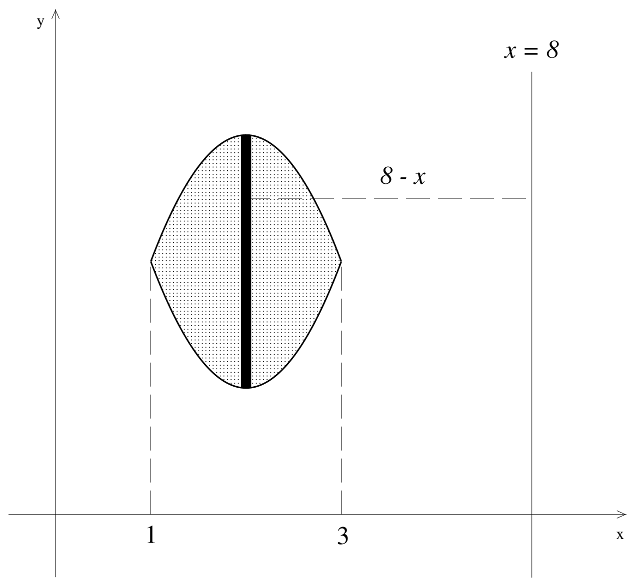
The integral to obtain the volume then becomes: $V = 2\pi\int_0^4 x(8-x^{3/2})\ dx$

$$V = 2\pi \int_0^4 x(8 - x^{3/2})\ dx = 2\pi \int_0^4 (8x - x^{5/2})\ dx$$
$$= 2\pi \left[4x^2 - \frac{2}{7}\, x^{7/2} \right]_0^4$$
$$= 2\pi \left[\left(64 - \frac{256}{7} \right) - 0 \right]$$
$$= \frac{384\pi}{7}$$

Therefore the volume is $\frac{384\pi}{7}$.

Example 2

Using the shell method, find the volume of the solid generated when the region bounded by $y = -(x-2)^2 + 3$ and $y = (x-2)^2 + 1$ is rotated about the line $x = 8$.



Once again, in order to get a shell, our element is vertical which means we do the problem in terms of x.

The radius is $8 - x\ldots$ the distance from the element to the axis of rotation.

The height of the shell is the length of the element which is: $(-(x-2)^2 + 3) - ((x-2)^2 + 1)$.

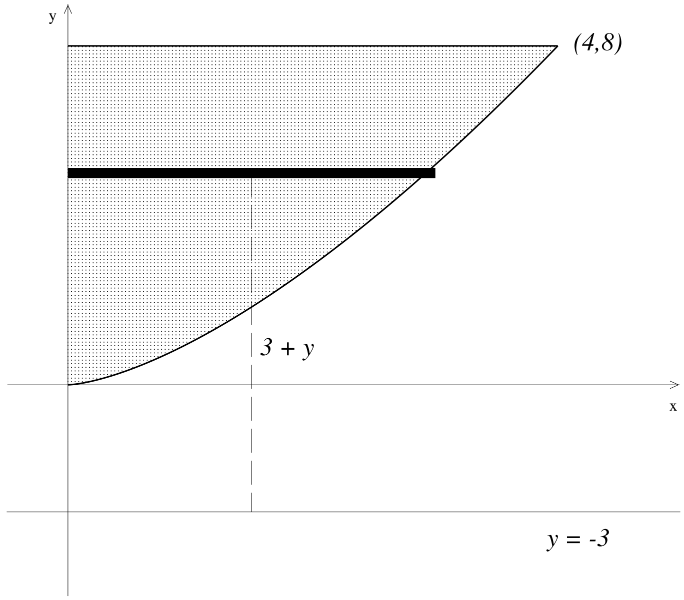
The integral which yields the volume then becomes:

$$V = 2\pi \int_1^3 (8 - x) \left[(-(x-2)^2 + 3) - ((x-2)^2 + 1) \right]\ dx$$

(The evaluation of the integral is left to the reader. It may look bad but is quite easy once everything is simplified–it's just the power rule.)

Example 3

Using the shell method, set up but do not evaluate an integral to find the volume of the solid generated when the region bounded by $y^2 = x^3$, $y = 8$ and $x = 0$ is rotated about the line $y = -3$.



Since our element must now be horizontal, the problem will be done in terms of y.

$$y^2 = x^3 \longrightarrow x = y^{2/3}$$

The intersection is still at $(4, 8)$ and because we are integrating in y our bounds will be 0 to 8.

The point where the element touches the curve is $(y^{2/3}, y)$ so the height of the shell is $y^{2/3}$.

The distance from the x-axis to the line $y = -3$ is 3 (distances are positive) and the distance from the x-axis to the element is y so the radius of the shell is $3 + y$.

The integral to obtain the volume is then:

$$V = 2\pi \int_0^8 (3 + y) \left(y^{2/3}\right) \, dy$$

To evaluate this simply distribute the $y^{2/3}$ and then use the power rule.

Chapter 11

Additional Applications of the Definite Integral

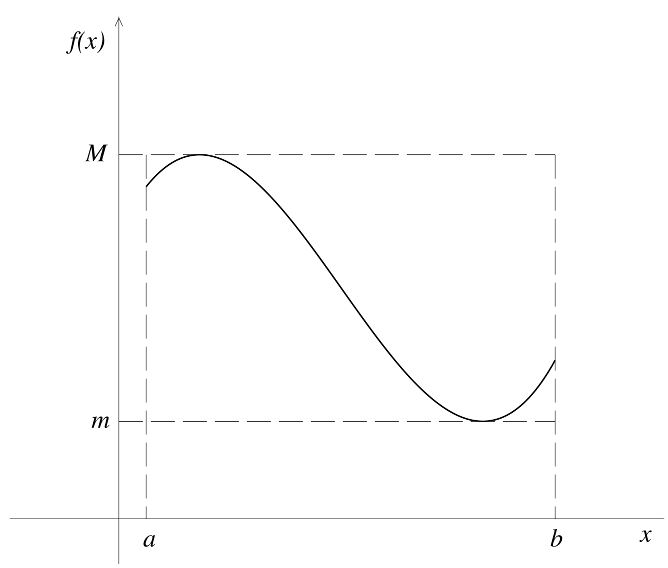
11.1 Average Value

11.1.1 Introduction

Finding the average value of a continuous function on an interval is yet another application of the definite integral. You've found averages many times. You add up the individual values and then divide by the number of values. Think about doing this for a continuous function on an interval. First you'd have to add up an infinite number of function values... that's bad enough... now you'd have to divide by the infinite number of values. Not possible in most instances. Instead we will use the definite integral–it's shown its usefulness many times for us in accumulating infinite numbers of objects. On our way to finding a method to calculate average value, we will come upon a very interesting intermediate result.

11.1.2 A nice intermediate result

Consider the following diagram. It shows a function f on a closed interval $[a, b]$. The absolute maximum function value on $[a, b]$ is labeled M and the absolute minimum function value on $[a, b]$ is labeled m.



Note that $M(b-a)$ is the area of the large rectangle bounded by $x=a$, $x=b$, $y=M$ and the x-axis. Similarly, $m(b-a)$ is the area of the smaller rectangle bounded by $x=a$, $x=b$, $y=m$ and the x-axis.

We know that the actual area under f is given by:

$$\int_a^b f(x)\ dx.$$

It's pretty clear that the area of the larger rectangle is larger than the actual area under the curve and the area under the curve is larger than the area of the smaller rectangle as denoted below:

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$$

At first glance this may seem obvious and unimportant but it is actually a very important result. It means that we can put a lower bound and an upper bound on a definite integral without ever integrating! Why is this so important? First, differentiation is easier than integration and all we need is differentiation to find the M and m. Secondly, there are some functions that cannot be directly antidifferentiated so we need a means to at least know in what interval the value lies. Also consider that many scientific and engineering problems do not require exact values of integrals but only require that the value of an integral is at least one value but no higher than another value. For example, in industry most machine parts are made to certain "tolerances". A piston in an engine must be big enough so that it does not rattle around in the cylinder but must not be so big that it does not fit. What is needed is a maximum and minimum value.

Using the relationship we just discovered, we can put an upper and lower bound on an integral by simply finding the absolute maximum and absolute minimum function values on the interval.

Example 1

Consider the function $f(x)=x^3-2x^2-1$ on $[0,3]$. Without integrating, find a closed interval which contains the value of $\int_0^3 f(x)\ dx$.

All we need to do if find the absolute maximum and absolute minimum value on the given interval.

We can do this by using the Extreme Value Theorem.

$$f'(x)=3x^2-4x$$

$f'(x)\ \exists\ \forall x$ and $f'(x)=0 \longrightarrow x=0$ or $x=1.333$.

Both of these critical numbers are in the interval $[0,3]$.

Now we find the function values at the endpoints and critical numbers.

$f(0)=-1$ and $f(3)=8$ and $f(1.333)=2.185$.

The absolute maximum is 8 and the absolute minimum is 2.185.

Multiplying each of these by $(3-0)$ which is our $(b-a)$ gives us the interval $[-6.556, 24]$.

$$\text{Therefore} \int_0^3 (x^3 - 2x^2 - 1)\ dx\ \in\ [-6.556, 24]$$

Let's return now to the average value of a function on an interval. We know that there is a function value (M)which, when multiplied by $(b-a)$ will yield the area of a rectangle whose area is always larger than the actual area. We also know there is a function value (m)which, when multiplied by $(b-a)$ will yield the area of a rectangle whose area is always smaller than the actual area. By the Intermediate Value Theorem, there must exist a function value that, when multiplied by $(b-a)$ will yield the area of a rectangle that is exactly the same as the actual area under the curve. This function value is called the "average value of f on $[a, b]$. The theorem which states this is called the Mean Value Theorem for Integrals. (Remember, we also had a Mean Value Theorem for Derivatives.)

The Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there exists a z in $[a, b]$ such that

$$f(z) = \frac{1}{b-a} \int_a^b f(x)\ dx.$$

This function value $f(z)$ is called the *average value* of f on $[a, b]$.

You need to take a moment to think about the amazing result of calculus. Since a function has an infinite number of function values on any given interval, you are actually finding the average of an infinite number of numbers! It is literally the average value…in the same way that 8 is the average of 7 and 9!

Example 2

Find the average value of $f(x) = 4 - x^2$ on $[0, 2]$.

$$\begin{aligned} f_{avg} &= \frac{1}{2-0} \int_0^2 (4 - x^2)\ dx \\ &= \frac{1}{2}\left[4x - \frac{1}{3}\ x^3\right]_0^2 \\ &= \frac{8}{3} \end{aligned}$$

Example 3

The temperature (in degrees Fahrenheit) in a certain city t hours after 9 a.m. was approximated by $T = 50 + 14\sin\frac{\pi t}{12}$. Find the average temperature during the period from 9 a.m. to 9 p.m.

$$\begin{aligned} T_{avg} &= \frac{1}{12-0}\int_{1}^{12}\left[50 + 14\sin\frac{\pi t}{12}\right]\,dt \\ &= \frac{1}{12}\left[50t - \frac{168}{\pi}\cos\frac{\pi t}{12}\right]_0^{12} \\ &= 58.913 \end{aligned}$$

Therefore, the average temperature was 58.913 degrees Fahrenheit.

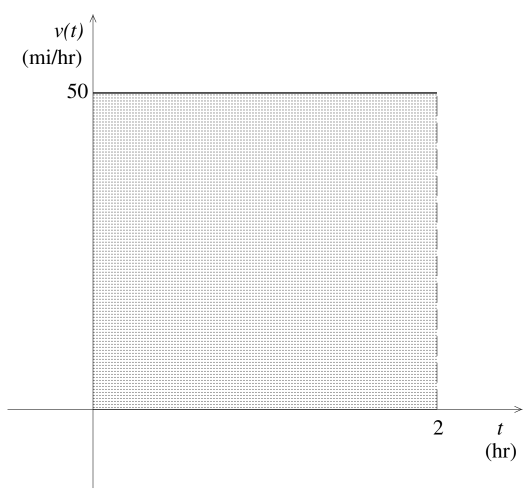
11.2 Total and Net Distance

11.2.1 Introduction

One of the more classic applications of the definite integral is the calculation of total distance traveled and net distance traveled. The definite integral is an accumulator. For areas, the definite integral accumulated the areas of infinite numbers of rectangles. For volumes it accumulated the volumes of infinite numbers of disks, washers or shells. We now continue our use of the definite integral as an accumulator–this time with distance.

11.2.2 Distance

Consider a car moving at a constant velocity of 50 miles per hour. If it moves at this velocity for two hours, it will travel 100 miles. Let's look at this problem in terms of area. In this case, the velocity is constant so the velocity function could be written as $v(t) = 50$. Below is a graph of this function on the interval $[0, 2]$. The area under $v(t) = 50$ between $t = 0$ and $t = 2$ is shaded.

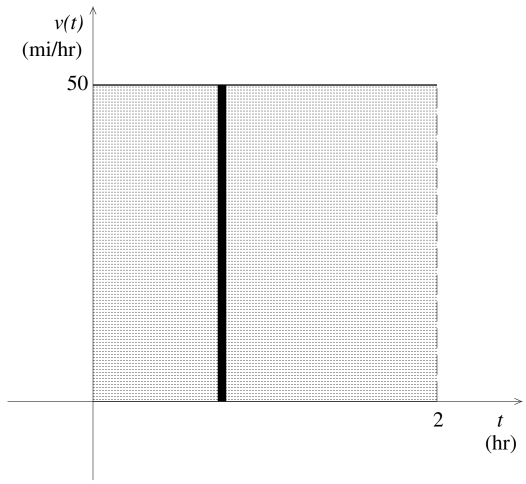


Notice that the shaded area is just a rectangle and its area is 100. This clearly represents the total distance the car traveled.

This area could also be expressed as a definite integral:

$$\int_0^2 50\ dt$$

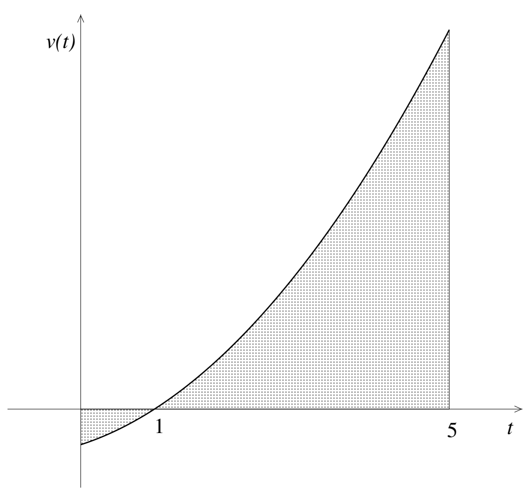
If we treated this as a area under a curve problem we would draw our element in as in the diagram below.



Now notice that the units on the height of our element are $\dfrac{miles}{hour}$ and the units on the width dt are hours. When we find the area of our element we multiply these two together and the hours divide out and leave us with miles! Amazing! Integrate a velocity function and get a distance. Now, we have to be careful…there is total distance and net distance. I got out of bed this morning and tonight I'll get back in my bed. My net distance for the day will be zero…even if I fly to Paris and back during the day.

Let's look at another distance problem in which the velocity function is not always positive. Remember, if velocity is negative, we are going backwards!

Consider the velocity function $v(t) = 3t^2 + 6t - 9$ on the interval $t = 0$ to $t = 5$. The function is graphed below and the area bounded by the function and the t-axis is shaded.

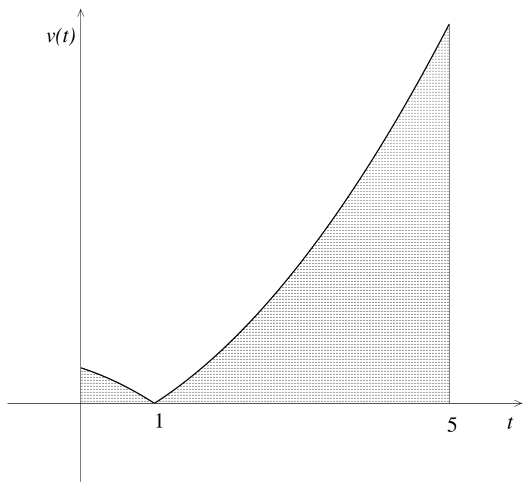


The shaded area is a measure of the distance the object travels from $t = 0$ to $t = 5$. If we integrated $v(t)$ on $[0, 5]$ we would get a number which is the area above the axis minus the area below the axis. Since the velocity curve is below the axis on $(0, 1)$, the object is moving backwards on $(0, 1)$. The small shaded region below the axis is a measure of how far it moved backward. The area above the axis is a measure of how far it moved forward. In this case the integral

$$\int_0^5 (3t^2 + 6t - 9)\ dt$$

represents the *net distance* the object travels.

What if we wanted to know the total distance the object traveled. We would need to add in the negative region. This can be accomplished by integrating the absolute value of the velocity function. Remember that taking the absolute value of a function simply reflects any portion of a curve which lies below the axis to above the axis. Below is the graph of $|v(t)|$.



If we integrated this curve, we would get the *total distance* traveled! Our integral for the total distance would be

$$\int_0^5 |3t^2 + 6t - 9|\ dt.$$

To evaluate this depends on whether or not we are using a calculator. If we are using a calculator we can just type in the function as it looks. If we are not using a calculator we must rewrite the integral without absolute value. We need to add in the area below the axis. Since it is a negative area,we will subtract it... thus making it positive. The total distance now becomes

$$\int_0^5 |3t^2 + 6t - 9|\ dt = -\int_0^1 (3t^2 + 6t - 9)\ dt + \int_1^5 (3t^2 + 6t - 9)\ dt.$$

If the velocity function is always above the axis, the net and total distance will be the same. If the velocity function ever goes below the axis, we need to use absolute value to total distance.

Net distance: $$\int_{t_1}^{t_2} v(t)\ dt$$

Total distance: $$\int_{t_1}^{t_2} |v(t)|\ dt$$

If $v(t) \geq 0\ \forall\ t\ \in (t_1, t_2)$ then

Total distance=Net distance

Example 4

The velocity of a particle is given by $v(t) = 3t^2 + 6t - 9$ and the particle is at position -5 when $t = 1$, find the position of the particle at $t = 3$.

This is a classic *initial position* problem. We know the particle is at -5 when $t = 1$ so we need to know how far the particle moves from $t = 1$ to $t = 3$. Our velocity function is above the axis for $t > 1$ so we can just add the distance traveled from $t = 1$ to $t = 3$ to the initial position.

$$-5 + \int_{1}^{3} (3t^2 + 6t - 9)\ dt = 27$$

Therefore, at $t = 3$ the particle is at position 27.

11.3 More on the Definite Integral as an Accumulator

11.3.1 General accumulator applications

Given a function that describes the rate of change in a quantity, its integral over a specific interval will yield the "total amount of change". If you integrate a rate function, you get an amount of change.

If $R(t)$ describes the rate of growth of a child in inches per year from birth to age 8, the definite integral

$$\int_{0}^{3} R(t)\ dt$$

represents the total growth in the child by the end of the third year. Note this it is not the height of the child…it's how much the child grew. To know how tall the child is at the end of three years, we would need to add this to the child's original height.

Think of integrating this function $R(t)$. If we drew an element in the region bounded by $t = 0$ and $t = 3$, the units on the height of the element would be inches/year and the units on the dt would be years. When we multiply these two to get the area of our ith rectangle, the "years" reduce and we are left with inches!

If $E(t)$ represents the rate of electricity usage in kilowatts/hour over a 24-hour period,

$$\int_0^7 E(t)\ dt$$

represents the total amount of electricity in kilowatts used in the first seven hours.

Notice that in the functions above $R(t)$ and $E(t)$, we did not use absolute value. That because we know these function are always greater than zero. If $R(t)$ was ever below the axis, the child would be shrinking! This is true of most acccumulator problems. Most of our rate functions are positive and we need not worry about absolute value to get total change. Velocity functions are the only ones we need to worry about. Velocity function will quite often be negative (below the axis) on some part of the interval in question.

Example 5

Shampoo drips from a crack in the side of a plastic bottle at a rate modeled by the function

$$Y(t) = \frac{t}{\sqrt{1+t^{3/2}}}$$

, where $Y(t)$ is in ounces per minute. If there are 32 ounces in the bottle at $t = 0$, how many ounces are left in the bottle after 5 minutes.

We will take the original amount of shampoo present initially (32 ounces) and subtract the total amount that leaks.

$$32 - \int_0^5 Y(t)\ dt = 26.937$$

Therefore, after 5 minutes there is 26.937 ounces of shampoo left in the bottle.

(This integral was evaluated using a calculator.)

Example 6

The number of home fires each day in a certain city increases as the temperature drops. The rate of home fires is modeled by

$$F(t) = 4\cos\left(\frac{t}{58} - 2\right) + 6$$

, for $0 \le t \le 365$, where midnight on January 1st corresponds to $t = 0$. How many home fires occurred during January?

Since there are 31 days in January, to get to total number of fires we use

$$\int_0^{31} F(t)\ dt = 166.242$$

Therefore there were 166.242 home fires.