Advanced Placement Calculus

Limits

1. The point $P(1,3)$ lies on the curve $y = 1 + x + x^2$.

   (a) If Q is the point $(x, 1 + x + x^2)$, find the slope of the secant line PQ for the following values of x:

x	2	1.5	1.1	1.01	1.001
Slope					

x	0	0.5	0.9	0.99	0.999
Slope					

   (b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(1,3)$.

   (c) Using the slope from part (b), find an equation of the tangent to the curve at $P(1,3)$.

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2. If a ball is thrown into the air with a velocity of 40 feet per second, its height after t seconds is given by $y = 40t - 16t^2$.

   (a) Find the average velocity for the time period beginning when $t = 2$ and lasting:

   0.5 seconds

   0.1 seconds

   0.05 seconds

   0.01 seconds

   0.001 seconds

   (b) Use the data from part (a) to estimate the instantaneous velocity when $t = 2$.

3. The displacement, in feet, of a particle moving in a straight line is given by $s = \frac{t^3}{6}$, where t is measured in secnods.

(a) Find the average velocity for the following time periods:

[1,3]

[1,2]

[1,1.5]

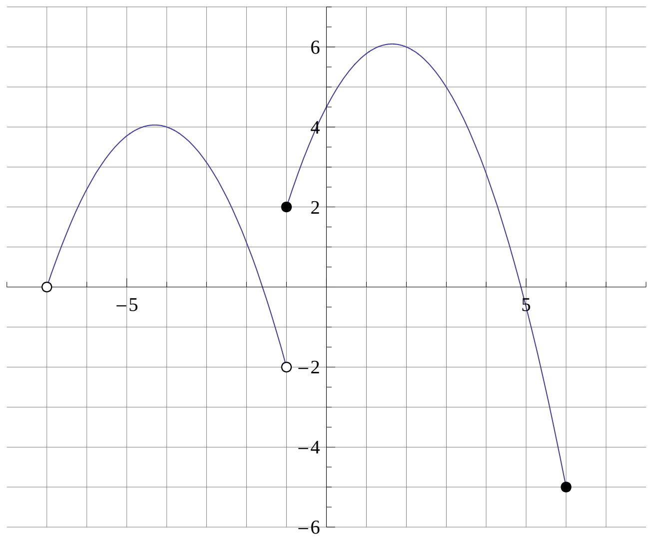
[1,1.1]

[1,1.01]

[1,1.001]

(b) Use the data from part (a) to estimate the instantaneous velocity when $t = 1$.

1. For the function f whose graph is given below, state the value of the given quantity, if it exists.



(a) $\lim_{x \to 4^+} f(x)$

(b) $\lim_{x \to 4^-} f(x)$

(c) $\lim_{x \to 4} f(x)$

(d) $\lim_{x \to -1^+} f(x)$

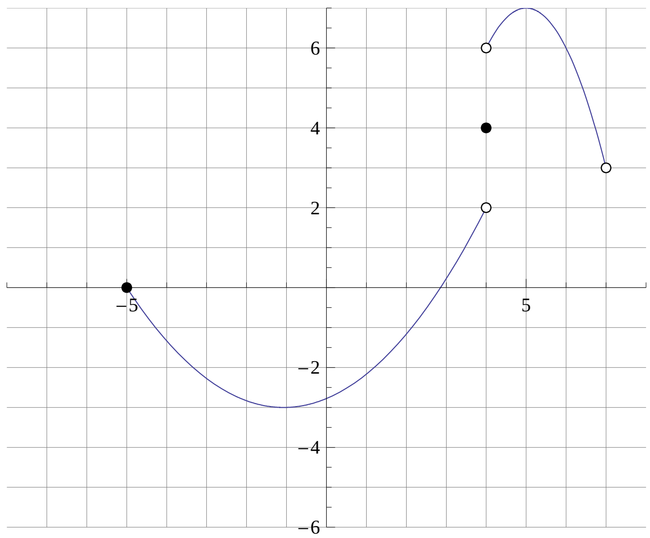
(e) $\lim_{x \to -1^-} f(x)$

(f) $\lim_{x \to -1} f(x)$

(g) $f(-1)$

(h) Describe the relationship between the limit of a function as $x \longrightarrow a$ and $f(a)$.

2. For the function f whose graph is given below, state the value of the given quantity, if it exists.



(a) $\lim_{x \to 4^+} f(x)$

(b) $\lim_{x \to 4^-} f(x)$

(c) $\lim_{x \to 4} f(x)$

(d) $f(4)$

(e) $\lim_{x \to -5^+} f(x)$

(f) $\lim_{x \to -5^-} f(x)$

(g) $\lim_{x \to -5} f(x)$

(g) $f(-5)$

3. Sketch the graph of $f(x) = \begin{cases} 1-x & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$ and find a-d below.

(a) $\lim_{x \to 0^-} f(x)$

(b) $\lim_{x \to 0^+} f(x)$

(c) $\lim_{x \to 0} f(x)$

(d) $f(0)$

Use your calculator's listing capabilities to evaluate the function at the given list of values. Use the information to estimate the value of the limit.

4. $g(x) = \dfrac{x-1}{x^3-1}$ with $x = \{0.2, 0.4, 0.6, 0.8, 0.9, 0.99, 0.999\}$ and $x = \{1.8, 1.6, 1.4, 1.2, 1.1, 1.01, 1.001\}$

Estimate $\lim_{x \to 1} g(x)$.

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5. $f(x) = \dfrac{1-\cos x}{x^2}$ with $x = \{1, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.01, 0.001\}$
and $x = \{-1, -0.5, -0.4, -0.3, -0.2, -0.1, -0.05, -0.01, -0.001\}$

Estimate $\lim_{x \to 0} f(x)$.

Estimate the following limits using lists of values.

6. $\lim_{x \to 5^+} \dfrac{1}{x-5}$

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7. $\lim_{x \to 3} \dfrac{1}{(x-3)^8}$

1. Evaluate: $\lim_{x\to 4}(5x^2-2x+3)$

2. Evaluate: $\lim_{x\to 2}(x^2+1)(x^2+4x)$

3. Evaluate: $\lim_{x\to -1}\dfrac{x-2}{x^2+4x-3}$

4. Evaluate: $\lim_{x\to -1}\sqrt{x^3+2x+7}$

5. Evaluate: $\lim_{t\to -2}(t+1)^9(t^2-1)$

6. Evaluate: $\lim_{w\to -2}\sqrt[3]{\dfrac{4w+3w^3}{3w+10}}$

7. What's wrong with the statement $\dfrac{x^2+x-6}{x-2}=x+3$?

8. Is there anything wrong with the statement $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$? Explain.

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9. Evaluate: $\lim_{x \to -3} \frac{x^2 - x + 12}{x + 3}$

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10. Evaluate: $\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1}$

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11. Evaluate: $\lim_{t \to 1} \frac{t^3 - 1}{t^2 - 1}$

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12. Evaluate: $\lim_{h \to 0} \frac{(h - 5)^2 - 25}{h}$

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13. Evaluate: $\lim_{h \to 0} \frac{(1 + h)^4 - 1}{h}$

14. Evaluate: $\lim_{x \to -2} \dfrac{x+2}{x^2 - x - 6}$

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15. Evaluate: $\lim_{t \to 9} \dfrac{9-t}{3-\sqrt{t}}$

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16. Evaluate: $\lim_{t \to 0} \dfrac{\sqrt{2-t}-\sqrt{2}}{t}$

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17. Evaluate: $\lim_{x \to 9} \dfrac{x^2 - 81}{\sqrt{x}-3}$

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18. Evaluate: $\lim_{t \to 0} \left[\dfrac{1}{t\sqrt{1+t}} - \dfrac{1}{t} \right]$

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19. Evaluate: $\lim_{x \to 0} \dfrac{x}{\sqrt{1+3x}-1}$

20. Evaluate: $\lim_{x \to 2} \frac{x - \sqrt{3x - 2}}{x^2 - 4}$

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21. Evaluate: $\lim_{x \to 4^-} \sqrt{16 - x^2}$

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22. Evaluate: $\lim_{x \to -4} |x + 4|$

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23. Evaluate: $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$

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24. Evaluate: $\lim_{x \to 2^+} \|x\|$

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25. Evaluate: $\lim_{x \to -2.4} \|x\|$

26. Evaluate: $\lim_{x \to 1^+} \sqrt{x^2 + x - 2}$

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27. The signum function, denoted by sgn is defined by: $sgn(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$. Sketch the graph of this function and then find the limits a-d listed below:

(a) $\lim_{x \to 0^+} sgn(x)$

(b) $\lim_{x \to 0^-} sgn(x)$

(c) $\lim_{x \to 0} sgn(x)$

(d) $\lim_{x \to 0^+} |sgn(x)|$

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28. Let $f(x) = \begin{cases} x^2 - 2x + 2 & \text{if } x < 1 \\ 3 - x & \text{if } x \geq 1 \end{cases}$. Find $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 1^-} f(x)$ and then $\lim_{x \to 1} f(x)$ if it exists.

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29. If n is an integer, find $\lim_{x \to n^-} \|x\|$ and $\lim_{x \to n^+} \|x\|$.

30. For what values of a will $\lim_{x \to a} \|x\|$ exist?

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31. Let $f(x) = \dfrac{x^2 - 1}{|x - 1|}$. Find $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1} f(x)$ if they exist.

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32. Evaluate: $\lim_{x \to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$

33. Given $f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ \frac{x}{2} + 1 & \text{if } x > 2 \end{cases}$, find $\lim_{x \to 2} f(x)$, if it exists.

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34. Given $f(x) = \begin{cases} 1 - x^2 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$, find $\lim_{x \to 1} f(x)$, if it exists.

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35. Given $f(x) = \begin{cases} 5x - 2 & \text{if } x \leq 3 \\ -2x + 5 & \text{if } x > 3 \end{cases}$, find $\lim_{x \to 3} f(x)$, if it exists.

36. Given $f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ 2 & \text{if } x = 2 \\ \dfrac{x}{2} & \text{if } x > 2 \end{cases}$, find $\lim_{x \to 2} f(x)$, if it exists.

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37. Given $f(x) = \begin{cases} x^2 - 2 & \text{if } x \leq 2 \\ 4 - x & \text{if } x > 2 \end{cases}$, find $\lim_{x \to 2} f(x)$, if it exists.

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38. Given $f(x) = \begin{cases} \dfrac{1}{x - 1} & \text{if } x < 1 \\ x^3 - 2x + 5 & \text{if } x \geq 1 \end{cases}$, find $\lim_{x \to 1} f(x)$, if it exists.

Find the vertical asymptotes (if any) of the following functions.

39. $f(x) = \dfrac{1}{x-3}$

40. $f(x) = \dfrac{x^2-25}{x-5}$

41. $f(x) = \dfrac{5x-1}{x+3}$

42. $f(x) = \dfrac{x^2-x-6}{x^2-3x}$

43. $f(x) = \dfrac{x^2 - x - 20}{x^2 + 9x + 20}$

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44. $f(x) = \dfrac{x^2 + 5x + 6}{x - 7}$

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45. $f(x) = \dfrac{x^2 + 5x + 6}{x + 3}$

1. Evaluate: $\lim_{x \to 0} \frac{\sin 4x}{x}$

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2. Evaluate: $\lim_{x \to 0} \frac{\sin 9x}{\sin 7x}$

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3. Evaluate: $\lim_{x \to 0} \frac{3x}{\sin 5x}$

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4. Evaluate: $\lim_{x \to 0} \frac{x^2}{\sin^2 3x}$

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5. Evaluate: $\lim_{x \to 0} \frac{x}{\cos x}$

6. Evaluate: $\lim_{x \to 0} \frac{1 - \cos 2x}{4x}$

7. Evaluate: $\lim_{x \to 0} \frac{\tan x}{2x}$

8. Evaluate: $\lim_{x \to 0} \frac{1 - \cos 8x}{\sin 3x}$

9. Evaluate: $\lim_{x \to 0} \frac{x^2 + 3x}{\sin x}$

1. How close to 3 do we have to take x so that $6x + 1$ is within 0.1 units of 19?

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2. How close to 2 do we have to take x so that $\dfrac{1}{x}$ is within 0.5 units of 0.5?

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3. How close to 1 do we have to take x so that x^2 is within $\dfrac{1}{2}$ units of 1?

Prove the following limits using the formal definition of limit.

4. $\lim_{x \to 2} (3x - 2) = 4$

5. $\lim_{x \to -1} (5x + 8) = 3$

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6. $\lim_{x \to 2} \frac{x}{7} = \frac{2}{7}$

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7. $\lim_{x \to -5} \left(4 - \frac{3x}{5}\right) = 7$

8. $\lim_{x \to a} x = a$

9. $\lim_{x \to 0} x^2 = 0$

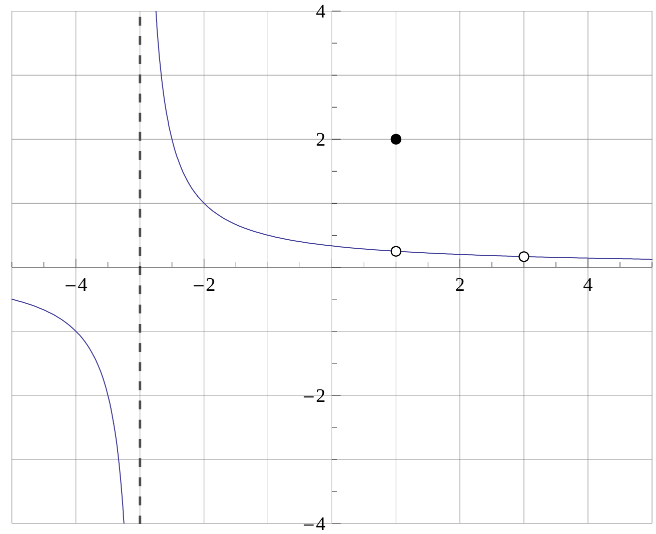
10. $\lim_{x \to 2} (x^2 - 4x + 5) = 1$

11. $\lim_{x \to -2} (x^2 - 1) = 3$

12. $\lim_{x \to 3} (x^2 + 5x + 6) = 30$

13. $\lim_{x \to 3} (x^2 + x - 4) = 8$

1. From the graph below, state where the function is discontinuous and state why (using the definition of continuity at a number.)



2. Use the definition of continuity at a number to show that $f(x) = x^4 - 5x^3 + 6$ is continuous at $x = 3$.

3. Use the definition of continuity at a number to show that $f(x) = 1 + \sqrt{x^2 - 9}$ is continuous at $x = 5$.

4. Use the definition of continuity at a number to show that $f(t) = \dfrac{\sqrt[3]{t}}{(t+1)^4}$ is continuous at $t = -8$.

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5. Explain why $f(x) = \dfrac{1}{(x-1)^2}$ is discontinuous at $x = 1$.

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6. Explain why $f(x) = \begin{cases} -\dfrac{1}{(x-1)^2} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$ is discontinuous at $x = 1$.

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7. Explain why $f(x) = \begin{cases} x^2 - 2 & \text{if } x \neq 3 \\ 5 & \text{if } x = 3 \end{cases}$ is discontinuous at $x = 3$.

8. For what value of the constant c will $f(x) = \begin{cases} cx+1 & \text{if } x \leq 3 \\ cx^2 - 1 & \text{if } x > 3 \end{cases}$ be continuous on $(-\infty, \infty)$?

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9. Find the values of the constants c and d that will make $h(x) = \begin{cases} 2x & \text{if } x < 1 \\ cx^2 + d & \text{if } 1 \leq x \leq 2 \\ 4x & \text{if } x > 2 \end{cases}$ continuous on $(-\infty, \infty)$.

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10. Use the definition of continuity at a number to determine if $f(x) = \dfrac{x^2 - 2x - 8}{x+2}$ is continuous for all real numbers. If it is not, determine if the discontinuity is removable or essestial. If the discontinuity is removable, redefine f so that f is continuous for all real numbers.

11. Use the definition of continuity at a number to determine if $f(x) = \dfrac{x^3 + 64}{x + 4}$ is continuous for all real numbers. If it is not, determine if the discontinuity is removable or essestial. If the discontinuity is removable, redefine f so that f is continuous for all real numbers.

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12. Use the definition of continuity at a number to determine if $f(x) = \dfrac{3 - \sqrt{x}}{9 - x}$ is continuous for all real numbers. If it is not, determine if the discontinuity is removable or essestial. If the discontinuity is removable, redefine f so that f is continuous for all real numbers.

13. Consider the following function: $f(x) = \begin{cases} 6+x & \text{if } x \leq -2 \\ 2-x & \text{if } -2 < x \leq 2 \\ 2x-1 & \text{if } x > 2 \end{cases}$

Determine at what values of x is the function discontinuous. Make sure you answer completely by using tests for continuity at a number.

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14. Determine at what values of x the following function is discontinuous. Make sure you answer completely by using tests for continuity at a number.

$$f(x) = \begin{cases} x^2-9 & \text{if } x \leq -3 \\ 9-x^2 & \text{if } -3 < x \leq 3 \\ 3x+2 & \text{if } x \geq 3 \end{cases}$$

15. Given $f(x) = x^3 - x^2 + x$, show that there is a number c such that $f(c) = 10$.

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16. Given $g(x) = x^5 - 2x^3 + x^2 + 2$, show that there is a number c such that $g(c) = -1$.

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17. Use the intermediate theorem to show that $f(x) = x^3 - 3x + 1$ has a zero on the interval $(0, 1)$ without finding the actual zeros.

Use the intermediate theorem to show that $f(x) = x^3 - x^2 + 2x - 1$ has a zero on the interval $(0, 1)$ without finding the actual zeros.

1. Evaluate: $\lim_{x\to\infty} \frac{1}{x\sqrt{x}}$

2. Evaluate: $\lim_{x\to\infty} \frac{x+4}{x^2-2x+5}$

3. Evaluate: $\lim_{x\to\infty} \frac{(1-x)(2+x)}{(1+2x)(2-3x)}$

4. Evaluate: $\lim_{x\to\infty} \frac{1}{3+\sqrt{x}}$

5. Evaluate: $\lim_{r\to\infty} \frac{r^4-r^2+1}{r^5+r^3-r}$

6. Evaluate: $\lim_{x\to\infty} \dfrac{\sqrt{1+4x^2}}{4+x}$

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7. Evaluate: $\lim_{x\to\infty} \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$

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8. Evaluate: $\lim_{x\to\infty} \left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$

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9. Evaluate: $\lim_{x\to\infty} \left(\sqrt{1+x}-\sqrt{x}\right)$

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10. Evaluate: $\lim_{x\to\infty} \sqrt{x}$

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11. Evaluate: $\lim_{x\to\infty} \left(x-\sqrt{x}\right)$

12. Evaluate: $\lim_{x \to -\infty} (x^3 - 5x^2)$

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13. Evaluate: $\lim_{x \to \infty} \frac{x^7 - 1}{x^6 + 1}$

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14. Evaluate: $\lim_{x \to \infty} \frac{\sqrt{x} + 3}{x + 3}$

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15. Evaluate: $\lim_{x \to -\infty} \frac{\sqrt{x} + 3}{x + 3}$

Find the vertical and horizontal asymptotes of each curve. You must show the limits you take to find these asymptotes. You *cannot* justify an asymptote without a limit!

16. $f(x) = \frac{x}{x + 4}$

17. $f(x) = \dfrac{x^3}{x^2 + 3x - 10}$

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18. $f(x) = \dfrac{x}{\sqrt[4]{x^4 + 1}}$

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19. $f(x) = \dfrac{x^2 - 25}{x - 5}$

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20. $f(x) = \dfrac{x - 3}{x + 2}$

1-4. Determine if the following statements are true or false. Include your reasoning.

1. $\lim\limits_{x\to 1} \dfrac{x-3}{x^2+2x-4} = \dfrac{\lim\limits_{x\to 1}(x-3)}{\lim\limits_{x\to 1}(x^2+2x-4)}$

2. If $f(x) > 1 \; \forall \, x$ and $\lim\limits_{x\to 0} f(x)$ exists, then $\lim\limits_{x\to 0} f(x) > 1$.

3. Let f be a function such that $\lim\limits_{x\to 0} f(x) = 6$. Then there exists a number δ such that if $0 < |x| < \delta$, then $|f(x) - 6| < 1$.

4. If p is a polynomial, then $\lim\limits_{x\to b} p(x) = p(b)$.

5. Evaluate: $\lim\limits_{x\to 4} \sqrt{x + \sqrt{x}}$

6. Evaluate: $\lim\limits_{t\to -1} \dfrac{t+1}{t^3 - t}$

7. Evaluate: $\lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$

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8. Evaluate: $\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 + 3x - 2}$

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9. Evaluate: $\lim_{t \to 0} \frac{17}{(t-6)^2}$

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10. Evaluate: $\lim_{s \to 16} \frac{4 - \sqrt{s}}{s - 16}$

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11. Evaluate: $\lim_{x \to 8^-} \frac{|x-8|}{x-8}$

12. Evaluate: $\lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x}$

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13. Evaluate: $\lim_{x \to \infty} \frac{1 + 2x - x^2}{1 - x + 2x^2}$

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14. Evaluate: $\lim_{x \to \infty} \frac{\sqrt{x^2 - 9}}{2x - 6}$

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15. Evaluate: $\lim_{x \to -\infty} \frac{\sqrt{x^2 - 9}}{2x - 6}$

16. Evaluate: $\lim_{x \to \infty} \left(\sqrt[3]{x} - \frac{x}{3} \right)$

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17. Prove: $\lim_{x \to 5} (7x - 27) = 8$

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18. Prove: $\lim_{x \to 2} (x^2 - 3x) = -2$

19. Consider the function: $f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (x-3)^2 & \text{if } x \geq 3 \end{cases}$.

(a) $\lim_{x \to 0^+} f(x)$

(b) $\lim_{x \to 0^-} f(x)$

(c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to 3^+} f(x)$

(e) $\lim_{x \to 3^-} f(x)$

(f) $\lim_{x \to 3} f(x)$

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20. Use the Intermediate Value Theorem to show that $2x^3 + x^2 + 2 = 0$ has a root on the interval $(-2, -1)$. Do *not* solve the equation!

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21. If f is continuous on $[3, 5]$ and $f(3) = -4$ and $f(5) = 6$, what conclusion can you make about $f(x) = 0$ based on the Intermediate Value Theorem?