Advanced Placement Calculus

The Derivative

Definition of Derivative

1. If $f(x) = 3x^2 - 5x$, find $f'(x)$ using the definition of derivative. Then find $f'(2)$ and use it to write an equation of a tangent to the parabola $y = 3x^2 - 5x$ at $(2, 2)$.

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2. If $F(x) = x^3 - 5x + 1$, find $F'(x)$ using the definition of derivative. Then find $F'(0)$ and use it to write an equation of a tangent to the curve $y = x^3 - 5x + 1$ at $(0, 1)$.

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3. A particle moves along a straight line with equation of motion $f(t) = t^2 - 6t - 5$ where position is measured in meters and time in seconds. Find the velocity at $t = 2$.

For the following functions, find the derivative using the definition of derivative.

4. $f(x) = 1 + x - 2x^2$

5. $f(x) = \dfrac{x}{2x-1}$

6. $f(x) = \dfrac{2}{\sqrt{3-x}}$

7. Consider $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$. This limit represents the derivative of some function at some number. Find the function and the number at which the derivative is being taken.

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8. Consider $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$. This limit represents the derivative of some function at some number. Find the function and the number at which the derivative is being taken.

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9. Consider $\lim_{t\to 0} \frac{\sin\left(\frac{\pi}{2}+t\right)-1}{t}$. This limit represents the derivative of some function at some number. Find the function and the number at which the derivative is being taken.

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10. Consider $\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$. This limit represents the derivative of some function at some number. Find the function and the number at which the derivative is being taken.

11. Given $f(x) = 5x + 3$, find $f'(x)$ using the definition of derivative.

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12. Given $f(x) = x^3 - x^2 + 2x$, find $f'(x)$ using the definition of derivative.

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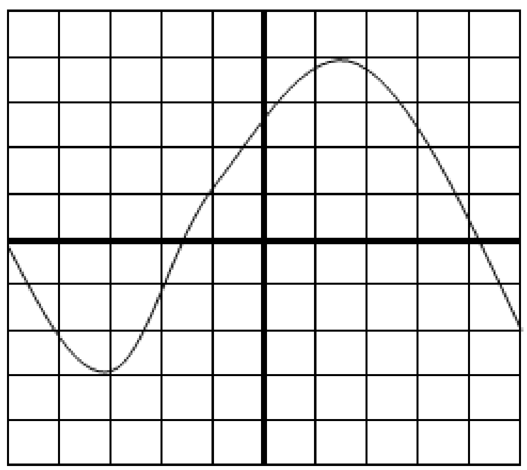
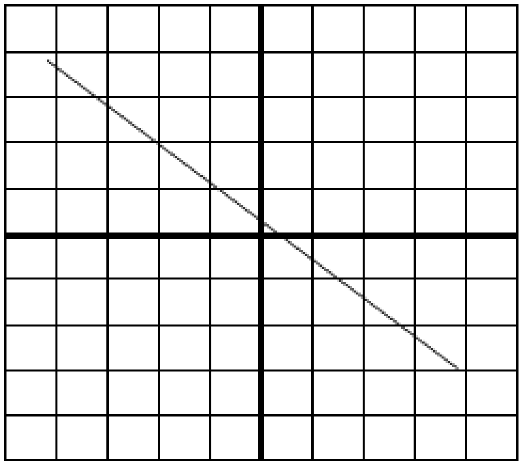
13. Given $G(x) = \sqrt{1 + 2x}$, find $G'(x)$ using the definition of derivative.

14. Given $f(x) = x^4$, find $f'(x)$ using the definition of derivative.

15. Use the sketch of f given below to estimate the following: $f'(-3)$, $f'(1.5)$, $f'(-1)$ and $f'(-4)$.



16. For the function graphed below, sketch the graph of $f'(x)$. Draw $f'(x)$ right on the same grid.



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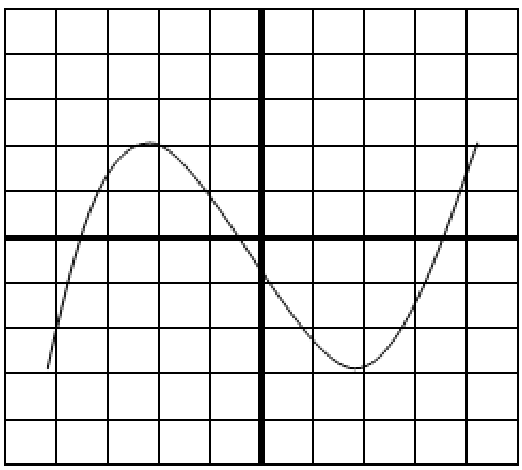
17. For the function graphed below, sketch the graph of $f'(x)$. Draw $f'(x)$ right on the same grid.



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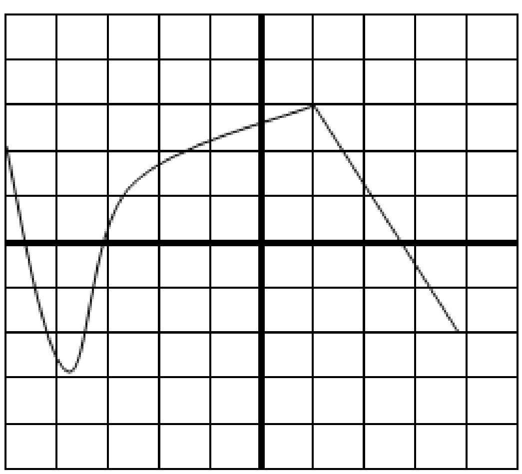
18. For the function graphed below, sketch the graph of $f'(x)$. Draw $f'(x)$ right on the same grid.



1. Using the definition of derivative, find the slope of the tangent line to the parabola $y = x^2 + 2x$ at the point $(-3, 3)$.

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2. Using the definition of derivative, find the slope of the tangent line to the parabola $y = 1 - 2x - 3x^2$ at the point $(-2, -7)$.

3. Using the definition of derivative, find the slope of the tangent line to the curve $y = \dfrac{1}{x^2}$ at $x = -2$.

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4. Using the definition of derivative, find the slope of the tangent line to the curve $y = \dfrac{2}{x+3}$ at $x = -1$.

5. If a ball is thrown vertically upward with a velocity of 40 feet per second, its height in feet after t seconds is given by $y = 40t - 16t^2$. Using the definition of derivative, find the velocity when $t = 2$.

6. The displacement in meters of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds. Find the average velocity on the the following intervals: $[3, 4]$, $[3.5, 4]$, $[4, 5]$ and $[4, 4.5]$. Then, using the definition of derivative, find the velocity when $t = 4$.

1. Given $f(x) = x^2 - 10x + 100$, find $f'(x)$.

2. Given $V(r) = \frac{4}{3}\pi r^3$, find $V'(r)$.

3. Given $F(x) = (16x)^3$, find $F'(x)$.

4. Given $Y(t) = 6t^{-9}$, find $Y'(t)$.

5. Given $g(x) = x^2 + \frac{1}{x^2}$, find $g'(x)$.

6. Given $h(x) = \dfrac{x+2}{x-1}$, find $h'(x)$.

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7. Given $G(s) = (s^2 + s + 1)(s^2 + 2)$, find $G'(s)$.

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8. Given $H(t) = \sqrt[3]{t}(t+2)$, find $H'(t)$.

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9. Given $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$, find $\dfrac{dy}{dx}$.

10. Given $y = \sqrt{5x}$, find $\frac{dy}{dx}$.

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11. Given $y = \frac{1}{x^4 + x^2 + 1}$, find $\frac{dy}{dx}$.

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12. Given $y = ax^2 + bx + c$ where a, b and c are constants, find $\frac{dy}{dx}$.

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13. Given $y = \frac{3t - 7}{t^2 + 5t - 4}$, find $\frac{dy}{dt}$.

14. Given $y = x + \sqrt[5]{x^2}$, find $\dfrac{dy}{dx}$.

15. Given $f(x) = x^{\sqrt{2}}$, find $f'(x)$.

16. Given $v = x\sqrt{x} + \dfrac{1}{x^2\sqrt{x}}$, find $\dfrac{dv}{dx}$.

17. Given $f(x) = \dfrac{x}{x + \dfrac{c}{x}}$, where c is a constant, find $f'(x)$.

18. Given $f(x) = \dfrac{x^5}{x^3 - 2}$, find $f'(x)$.

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19. Write an equation of a line tangent to $y = \dfrac{x}{x-3}$ at the point $(6, 2)$.

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20. Write an equation of a line tangent to $y = x^{5/2}$ at the point $(4, 32)$.

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21. Write an equation of a line tangent to $y = x + \dfrac{4}{x}$ at the point where $x = 2$.

22. Find the equations of the tangent lines to the curve $y = \dfrac{x-1}{x+1}$ that are parallel to the line $x - 2y = 1$.

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23. At what point on the curve $y = x\sqrt{x}$ is the tangent line parallel to $3x - y + 6 = 0$?

24. For what values of x does the graph of $f(x) = 2x^3 - 3x^2 - 6x + 87$ have a horizontal tangent?

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25. Find the points on the curve $y = x^3 - x^2 - x + 1$ where the tangent is horizontal.

26. Find the equations of both lines that pass through the point $(2, -3)$ that are tangent to the curve $y = x^2 + x$.

27. Write an equation of the normal to $y = 1 - x^2$ at the point $(2, -3)$.

28. Write an equation of the normal to $y = \sqrt[3]{x}$ at the point $(-8, -2)$.

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29. At what point on the curve $y = x^4$ does the normal line have slope 16?

1. Evaluate: $\lim_{x \to 0} (x^2 + \cos x)$

2. Evaluate: $\lim_{x \to \pi/3} (\sin x - \cos x)$

3. Evaluate: $\lim_{t \to \pi/4} \frac{\sin 5t}{t}$

4. Evaluate: $\lim_{x \to 0} \frac{\sin(\cos x)}{\sec x}$

5. Evaluate: $\lim_{x \to \pi/4} \frac{\sin x}{3x}$

6. Evaluate: $\lim_{x \to \pi/4} \frac{\tan x}{4x}$

7. Evaluate: $\lim_{x \to 0} \frac{\sin^2 x}{x}$

8. Evaluate: $\lim_{x \to 0} \frac{\tan 3x}{3 \tan 2x}$

9. Given $y = \cos x - 2 \tan x$, find $\frac{dy}{dx}$.

10. Given $y = \sin x + \cos x$, find $\frac{dy}{dx}$.

11. Given $y = x \csc x$, find $\dfrac{dy}{dx}$.

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12. Given $y = \csc x \cot x$, find $\dfrac{dy}{dx}$.

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13. Given $y = \dfrac{\sin x}{1 + \cos x}$, find $\dfrac{dy}{dx}$.

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14. Given $y = \dfrac{\tan x}{x}$, find $\dfrac{dy}{dx}$.

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15. Given $y = \dfrac{x}{\sin x + \cos x}$, find $\dfrac{dy}{dx}$.

16. Given $f(x) = x^{-3} \sin x \tan x$, find $f'(x)$.

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17. Given $g(x) = \dfrac{x^2 \tan x}{\sec x}$, find $g'(x)$.

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18. Write an equation of the line tangent to $y = x \cos x$ at the point $(\pi, -\pi)$.

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19. Write an equation of the line tangent to $f(x) = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

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20. Write an equation of the line tangent to $g(x) = 2 \sin x$ at the point where $x = \frac{\pi}{6}$.

21. For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

1. Given $y = u^2$ and $u = x^2 + 2x + 3$, find $\dfrac{dy}{dx}$.

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2. Given $y = u^2 - 2u + 3$ and $u = 5 - 6x$, find $\dfrac{dy}{dx}$.

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3. Given $y = u^3$ and $u = x + \dfrac{1}{x}$, find $\dfrac{dy}{dx}$.

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4. Given $y = u - u^2$ and $u = \sqrt{x} + \sqrt[3]{x}$, find $\dfrac{dy}{dx}$.

5. Given $F(x) = (x^2 + 4x + 6)^2$, find $F'(x)$.

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6. Given $G(x) = (x^3 - 5x)^4$, find $G'(x)$.

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7. Given $g(x) = (3x - 2)^{10}(5x^2 - x + 1)^{12}$, find $g'(x)$.

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8. Given $f(t) = (2t^2 - 6t + 1)^{-8}$, find $f'(t)$.

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9. Given $g(x) = \sqrt{x^2 - 7x}$, find $g'(x)$.

10. Given $h(t) = \left(t - \frac{1}{t}\right)^{3/2}$, find $h'(t)$.

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11. Given $f(y) = \left(\frac{y-6}{y+7}\right)^3$, find $f'(y)$.

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12. Given $s(t) = \sqrt[4]{\frac{t^3+1}{t^3-1}}$, find $s'(t)$.

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13. Given $f(z) = \frac{1}{\sqrt[5]{2z-1}}$, find $f'(z)$.

14. Given $y = \tan 3x$, find $\dfrac{dy}{dx}$.

15. Given $f(x) = \cos(x^3)$, find $f'(x)$.

16. Given $f(x) = \cos^3 x$, find $f'(x)$.

17. Given $y = \left(1 + \cos^2 x\right)^6$, find $\dfrac{dy}{dx}$.

18. Given $y = \cos(\tan x)$, find $\dfrac{dy}{dx}$.

19. Given $f(x) = \sec^2 2x - \tan^2 2x$, find $f'(x)$.

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20. Given $y = \csc\left(\frac{x}{3}\right)$, find $\frac{dy}{dx}$.

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21. Given $y = \sin^3 x + \cos^3 x$, find $\frac{dy}{dx}$.

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22. Given $p(x) = \sin\left(\frac{1}{x}\right)$, find $p'(x)$.

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23. Given $y = \frac{1 + \sin x}{1 - \sin x}$, find $\frac{dy}{dx}$.

24. Given $y = \tan^2 x^2$, find $\dfrac{dy}{dx}$.

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25. Given $y = \sqrt{x + \sqrt{x}}$, find $\dfrac{dy}{dx}$.

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26. Write an equation of the tangent to the curve $y = (x^3 - x^2 + x - 1)^{10}$ at the point $(1, 0)$.

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27. Write an equation of the tangent to the curve $f(x) = \dfrac{8}{\sqrt{4 + 3x}}$ at the point $(4, 2)$.

28. Find all the points on the graph of the function $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.

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29. Suppose that $F(x) = f(g(x))$ and $g(3) = 6,\ g'(3) = 4,\ f'(3) = 2$ and $f'(6) = 7$. Find $F'(3)$.

Additional Chain Rule Problems

1. Given $h(x) = f(g(x))$ and $f(2) = 5,\ f'(2) = 6,\ g(3) = 2$ and $g'(3) = 5$, find $h'(3)$.

2. Given $p(x) = q(r(x))$ and $q(-2) = 5,\ q'(3) = -4,\ r'(8) = -2$ and $r(8) = 3$, find $p'(8)$.

3. If $t(p) = u(v(w(p)))$ and $u'(4) = 12,\ v'(-3) = 2,\ v(-3) = 4,\ w'(5) = 6$ and $w(5) = -3$, find $t'(5)$.

4. Given $y = x^3 + \cos x^2$, find $\dfrac{dy}{dx}$.

5. Given $f(x) = \sin\left(\cos x^3\right)$, find $f'(x)$.

6. Given $g(x) = \sec^8(5x^3 - 17x)$, find $g'(x)$.

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7. Given $y = \csc^2\left(\cos^2 x\right)$, find $\dfrac{dy}{dx}$.

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8. Given $h(x) = \sqrt{x-1} + \sqrt{x+1}$, find $h'(x)$.

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9. Given $y = x^2 \tan\left(\dfrac{1}{x}\right)$, find $\dfrac{dy}{dx}$.

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10. Given $y = \dfrac{1}{\sqrt{\cos x}}$, find $\dfrac{dy}{dx}$.

11. Given $p = \sqrt[3]{\dfrac{x-3}{2x+5}}$, find $\dfrac{dp}{dx}$.

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12. Given $w = \dfrac{\sqrt{v}+1}{v^2+1}$, find $\dfrac{dw}{dv}$.

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13. Given $y = \cos(\sin(\tan x))$, find $\dfrac{dy}{dx}$.

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14. Given $f(x) = \cos\left(\sqrt{\tan^3 x}\right)$, find $f'(x)$.

15. Given $f(x) = [g(x)]^n$, find $f'(x)$.

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16. Given $g(x) = f(\tan x)$, find $g'(x)$.

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17. Given $h(x) = f\left(\sec^4 x\right)$, find $h'(x)$.

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18. Given $h(x) = f(x) \cdot [g(x)]^5$, find $h'(x)$.

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19. Given $g(x) = f(g(\sin x))$, find $g'(x)$.

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20. Given $h(x) = f\left([g(x)]^2\right)$, find $h'(x)$.

21. If $f(x) = g(x) \cdot h(x)$, and $g(5) = -3,\ g'(5) = 6,\ h(5) = 3$ and $h'(5) = -2$, find $f'(5)$, if possible. If not possible, tell what information is needed.

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22. If $f(x) = g(h(x))$, and $g(5) = -3,\ g'(5) = 6,\ h(5) = 3$, and $h'(5) = -2$, find $f'(5)$, if possible. If not possible, tell what information is needed.

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23. If $f(x) = \dfrac{g(x)}{h(x)}$, and $g(5) = -3,\ g'(5) = 6,\ h(5) = 3$, and $h'(5) = -2$, find $f'(5)$, if possible. If not possible, tell what information is needed.

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24. If $f(x) = [g(x)]^3$, and $g(5) = -3,\ g'(5) = 6,\ h(5) = 3$, and $h'(5) = -2$, find $f'(5)$, if possible. If not possible, tell what information is needed.

25. Given $f(x) = g(x^2 + 4x)$, find $f'(x)$.

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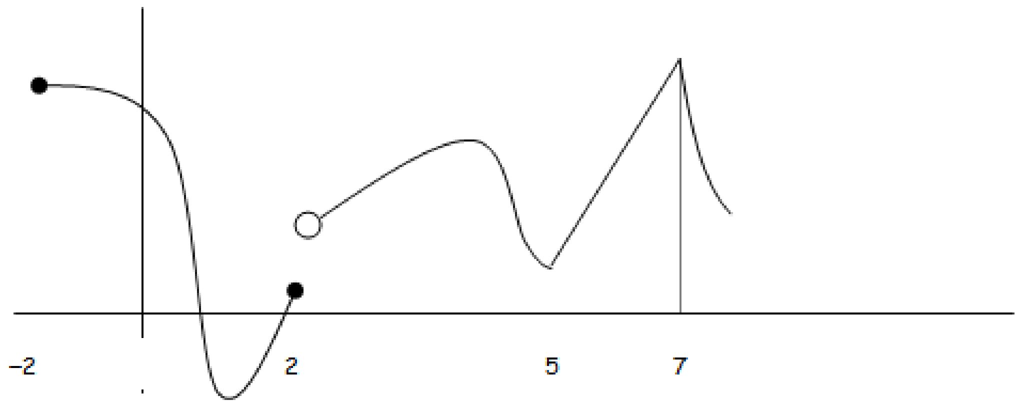
26. Given the information in the table and given the following functions, complete the table below. You may not be able to fill in every box.

$$g(x) = f(x) - 2$$
$$r(x) = f(-3x)$$
$$h(x) = 2f(x)$$
$$s(x) = f(x + 2)$$

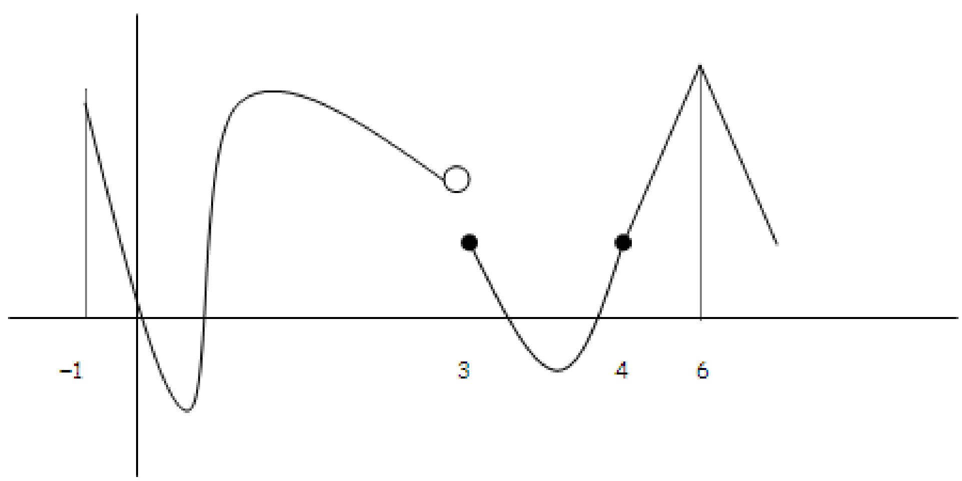
x	-2	-1	0	1	2	3
$f'(x)$	-2	2/3	-1/3	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

Differentiability

1. The graph of f is given below. State the x-values at which f is not differentiable and give a reason based on the definition of differentiability at a number.



2. The graph of f is given below. State the x-values at which f is not differentiable and give a reason based on the definition of differentiability at a number. Also state the x-values at which f is not continuous and give a reason based on the definition of continuity at a number.



3. Show that $f(x) = |x - 6|$ is not differentiable at $x = 6$.

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4. Where is the greatest integer function $f(x) = \|x\|$ not differentiable?

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5. Where and why is the following function given below not continuous? Where and why is it not differentiable?

$$f(x) = \begin{cases} \dfrac{x^3 - x}{x^2 + x} & \text{if } x < 1 \text{ but } x \neq 0 \\ 0 & \text{if } x = 0 \\ 1 - x & \text{if } x \geq 1 \end{cases}$$

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6. Given $f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x - 4 & \text{if } x > 0 \end{cases}$, find $f'(x)$ and tell where (if anywhere) the derivative does not exist.

1. Given $f(x) = x^4 - 3x^2 + 16x$, find $f'(x)$ and $f''(x)$.

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2. Given $h(x) = \sqrt{x^2 + 1}$, find $h'(x)$ and $h''(x)$.

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3. Given $F(s) = (3s + 8)^8$, find $F'(s)$ and $F''(s)$.

4. Given $y = \dfrac{x}{1-x}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

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5. Given $y = (1-x^2)^{3/4}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

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6. Given $H(t) = \tan^3(2t-1)$, find $H'(t)$ and $H''(t)$.

7. Given $f(x) = 2\cos x + \sin^2 x$, find $f'(x)$ and $f''(x)$.

8. Given $f(x) = \sqrt{5x - 1}$, find $f'''(x)$.

9. Given $f(x) = \dfrac{1}{\sqrt{2 - 3x}}$, find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$.

10. Given $f(\theta) = \cot\theta$, find $f'''\left(\frac{\pi}{6}\right)$.

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11. If the position function of a particle is given by $s(t) = t^3 - 3t$, find its velocity and acceleration functions, then find the acceleration when the velocity is zero.

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12. If the position function of a particle is given by $s(t) = At^2 + Bt + C$, find its velocity and acceleration functions, then find the acceleration when the velocity is zero.

Problems 1 - 6. State whether the following statements are true or false.

1. If f is continuous at a, then f is differentiable at a.

2. If f is differentiable at a, then f is continuous at a.

3. If f and g are differentiable, $\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g'(x)$.

4. If f is differentiable, then $\frac{d}{dx}\left[\sqrt{f(x)}\right] = \frac{f'(x)}{2\sqrt{f(x)}}$

5. If $g(x) = x^5$, then $\lim_{x\to 2} \frac{g(x) - g(2)}{x - 2} = 80$

6. An equation of the tangent to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

7. Given $f(x) = x^3 + 5x + 4$, find $f'(x)$ using the definition of derivative.

8. Given $f(x) = \sqrt{3-5x}$, find $f'(x)$ using the definition of derivative.

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9. Given $y = (x+2)^8(x+3)^6$, find $\dfrac{dy}{dx}$.

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10. Given $y = \dfrac{x}{\sqrt{9-4x}}$, find $\dfrac{dy}{dx}$.

11. Given $f(x) = \dfrac{x}{8-3x}$, find $f'(x)$.

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12. Given $y = \sqrt[5]{x \tan x}$, find $\dfrac{dy}{dx}$.

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13. Given $y = \dfrac{(x-1)(x-4)}{(x-2)(x-3)}$, find $\dfrac{dy}{dx}$.

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14. Given $g(x) = \tan\left(\sqrt{1-x}\right)$, find $g'(x)$.

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15. Given $y = \sin\left(\tan\left(\sqrt{1-x^2}\right)\right)$, find $\dfrac{dy}{dx}$.

16. Given $h(x) = \cot(3x^2 + 5)$, find $h'(x)$.

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17. Given $y = \cos^2(\tan x)$, find $\frac{dy}{dx}$.

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18. Given $f(x) = \frac{1}{(2x-1)^5}$, find $f''(0)$.

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19. Write an equation of a tangent to the curve $y = \frac{x}{x^2 - 2}$ at the point $(2, 1)$.

20. Write an equation of a tangent to the curve $f(x) = \tan x$ at the point $\left(\frac{\pi}{3}, \sqrt{3}\right)$.

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21. At what points on the curve $y = \sin x + \cos x$ where $x \in [0, 2\pi]$, is the tangent line horizontal?

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22. Suppose that $h(x) = f(x)g(x)$ and $F(x) = f(g(x))$ where $f(2) = 3$, $g(2) = 5$, $g'(2) = 4$, $f'(2) = -2$ and $f'(5) = 11$. Find $h'(2)$ and $F'(2)$.

23. Given $f(x) = x^2 g(x)$, find $f'(x)$ in terms of $g'(x)$.

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24. Given $f(x) = [g(x)]^2$, find $f'(x)$ in terms of $g'(x)$.

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25. Given $f(x) = g(g(x))$, find $f'(x)$ in terms of $g'(x)$.

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26. Given $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)}$, find $h'(x)$ in terms of $f'(x)$ and $g'(x)$.

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27. Express the following as a derivative and then evaluate: $\lim_{h \to 0} \dfrac{(2+h)^6 - 64}{h}$.