Advanced Placement Calculus

Applications of the Derivative I

Rectilinear Motion
Implicit Differentiation
Related Rates
Linearizations
Additional Local Linearity Problems
L'Hopital's Rule
Review

1. A particle is moving along a horizontal line according to the position function $s(t) = t^3 - 12t^2 + 36t - 24$ where $t \geq 0$. Determine the intervals of time when the particle is moving to the right and when it is moving to the left. Also determine the instant when the particle reverses direction.

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2. The position of an object dropped from the Arch is given by $s(t) = -16t^2 + 638$. If the Arch is 638 feet tall, how long will it take the object to reach the ground? How fast will the object be going at impact?

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3. An object is thrown vertically upward from the ground with an initial velocity of 30 feet per second. Its position is given by $s(t) = -16t^2 + 30t$. How high will the object go? With what velocity will it strike the ground?

4. The position of a stone thrown vertically upward from the ground is given by $x(t) = -16t^2 + 32t$ where s is measured in feet and t in seconds. Find (a) the average velocity on the interval $(.5, .75)$, (b) the instantaneous velocity at .5 seconds and .75 seconds, (c) the speed at .5 seconds and .75 seconds, (d) how many seconds it will take the stone to reach the highest point, (e) how high the stone will go, (f) how many seconds it will take to reach the ground, and (g) the instantaneous velocity of the stone when it hits the ground.

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5. A billiard ball is hit and travels in a straight line. If s centimeters is the distance of the ball from it initial position at t seconds, then $s = 100t^2 + 100t$. If the ball hits a cushion that is 39 centimeters from its initial position, at what velocity does it hit the cushion?

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6. A ball is thrown vertically upward from the top of a building 112 feet high. Its position above ground level is given by $s = -16t^2 + 96t + 112$. How high will the ball go? How long will it take to reach its maximum height? How long will it take the ball to hit the ground? How fast will the ball be going when it hits the ground?

1. Given $x^2 + 3x + xy = 5$, find $\dfrac{dy}{dx}$.

2. Given $2y^2 + xy = x^2 + 3$, find $\dfrac{dy}{dx}$.

3. Given $x^2 - xy + y^3 = 8$, find $\dfrac{dy}{dx}$.

4. Given $2y^2 + \sqrt[3]{xy} = 3x^2 + 17$, find $\dfrac{dy}{dx}$.

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5. Given $x^4 + y^4 = 16$, find $\dfrac{dy}{dx}$.

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6. Given $\dfrac{y}{x-y} = x^2 + 1$, find $\dfrac{dy}{dx}$.

7. Given $x\sqrt{1+y} + y\sqrt{1+2x} = 2x$, find $\dfrac{dy}{dx}$.

8. Given $\cos(x-y) = y \sin x$, find $\dfrac{dy}{dx}$.

9. Given $xy = \cot(xy)$, find $\dfrac{dy}{dx}$.

10. Given $y^4 + x^2y^2 + x^4y = y + 1$, find $\dfrac{dy}{dx}$.

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11. Given $x\,[f(x)]^3 + xf(x) = 6$ and $f(3) = 1$, find $f'(3)$.

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12. Given $[g(x)]^2 + 12x = x^2g(x)$ and $g(4) = 12$ find $g'(4)$.

13. Find an equation of the tangent to the curve $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ at the point $\left(-5, \dfrac{9}{4}\right)$.

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14. Find an equation of a tangent to the curve $y^2 = x^3(2 - x)$ at the point $(1, 1)$.

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15. Find an equation of a tangent to the curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point $(3, 1)$.

16. Two curves are called "orthogonal" if any any point the tangents to the curves are perpendicular. Prove that $2x^2 + y^2 = 3$ and $x = y^2$ are orthogonal.

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17. Find all the points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

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18. Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?

1. If V is the volume of a cube with edge length x, find $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$.

2. If $xy = 1$ and $\frac{dx}{dt} = 4$, find $\frac{dy}{dt}$ when $x = 2$.

3. A spherical snowball is melting in such a way that its volume is decreasing at a rate of 1 cubic centimeter per minute. At what rate is the diameter decreasing when the diameter is 10 centimeters.

4. A street light is at the top of a 15-foot pole. A man 6 feet tall walks away from the pole with a speed of 5 feet per second along a straight path. How fast is his shadow lengthening when he is 40 feet from the pole? How fast is the tip of his shadow moving when he is 40 feet from the pole?

5. A plane flying horizontally at an altitude of 1 mile and a speed of 500 miles per hour passes directly over a radar station. Find the rate at which the distance from the plane to the station (line-of-sight distance) is increasing when it is 2 miles (horizontally) from the station?

6. Two cars start moving from the same point. One travels south at 60 miles per hour and the other travels west at 25 miles per hour. At what rate is the distance between the cars increasing two hours later?

7. At noon, ship A is 100 kilometers west of ship B. Ship A is sailing south at 35 kilometers per hour and ship B is sailing north at 25 kilometers per hour. How fast is the distance between the ships increasing at 4 p.m.?

8. The altitude of a triangle is increasing at a rate of 1 centimeter per minute while the area of the triangle is increasing at a rate of 2 square centimeters per minute. At what rate is the base of the triangle changing when the altitude is 10 centimeters and the area is 100 square centimeters?

9. Water is leaking out of an inverted conical tank at a rate of 10,000 cubic centimeters per minute at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 meters and the diameter at the top is 4 meters. If the water level is rising at a rate of 20 centimeters per minute when the height of the water is 2 meters, find the rate at which water is being pumped into the tank.

10. Gravel is being dumped from a conveyor belt at a rate of 30 cubic feet per minute and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile changing when the pile is 10 feet high?

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11. A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 feet per second, how fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$ radians?

Linearizations

1. Linearize $f(x) = x^3$ at $x = 1$.

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2. Linearize $f(x) = \dfrac{1}{\sqrt{2+x}}$ at $x = 0$.

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3. Linearize $f(x) = \dfrac{1}{x}$ at $x = 4$.

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4. Linearize $f(x) = \sqrt[3]{x}$ at $x = -8$.

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5. Verify that the linearization of $\sqrt{1+x}$ is $1 + \dfrac{1}{2}x$ at $x = 0$.

6. Verify that the linearization of $\sin x$ is x at $x = 0$.

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7. Verify that the linearization of $\dfrac{1}{(1+2x)^4}$ is $1 - 8x$ at $x = 0$.

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8. Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $x = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. (Hint: Once you have your linearization, you'll need to evaluate it at $x = .1$ and then at $x - .01$. Can you see why?)

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9. Find the linear approximation of $g(x) = \sqrt[3]{1+x}$ at $x = 0$ and use it to approximate $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$.

Local Linearity Additional Problems

In problems 1-6, you need to find a linearization that will replace the given function over intervals that include the given value of x. To make you calculations as simple as possible, do not linearize at x_o but at some integer near x_o.

1. $f(x) = x^2 + 2x,\ x_o = 0.1$

2. $f(x) = x - 1,\ x_o = 0.6$

3. $f(x) = 2x^2 + 4x - 3,\ x_o = -0.9$

4. $f(x) = \sqrt[3]{x},\ x_o = 8.51$

5. $f(x) = \dfrac{x}{x+1},\ x_o = 1.3$

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6. $f(x) = \sqrt{x^2+9},\ x_o = -3.8$

Problems 7-10. Find the linearization of the given function at the given x-value.

7. $f(x) = x^4,\ x = 1$

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8. $f(x) = \sin x,\ x = 0$

9. $f(x) = \tan x$ at $x = \dfrac{\pi}{4}$

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10. $f(x) = (1 + x)^k$ at $x = 0$. (k is a constant)

Use an appropriate linearization to approximate the value of the given quantity. Verify your approximation with a direct calculator calculation.

11. $(3.02)^4$

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12. $\sqrt{24}$

13. $\sqrt[3]{26}$

14. $\sin 0.1$

1. Evaluate: $\lim_{x \to 2} \frac{x-2}{x^2-4}$

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2. Evaluate: $\lim_{x \to -1} \frac{x^6-1}{x^4-1}$

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3. Evaluate: $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$

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4. Evaluate: $\lim_{x \to 0} \frac{\sin x}{x^3}$

5. Evaluate: $\lim_{x \to 0} \frac{\tan x}{x + \sin x}$

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6. Evaluate: $\lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}}$

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7. Evaluate: $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

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8. Evaluate: $\lim_{x \to 0} \frac{x + \sin 3x}{x - \sin 3x}$

9. Evaluate: $\lim_{x \to 9} \frac{x^2 - 81}{\sqrt{x} - 3}$

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10. Evaluate: $\lim_{t \to 0} \frac{\sqrt{2 - t} - \sqrt{2}}{t}$

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11. Evaluate: $\lim_{x \to -3} \frac{x^2 - x - 12}{x + 3}$

1. Given $\sqrt[3]{x} - \sqrt[3]{y} = 1$, find $\dfrac{dy}{dx}$.

2. If a and b are constants and $\dfrac{x^2}{a} - \dfrac{y^4}{b} = 1$, find $\dfrac{dy}{dx}$.

3. Given $2x^3y + 3xy^3 = 5$, find $\dfrac{dy}{dx}$.

4. Given $x = \sin(x+y)$, find $\dfrac{dy}{dx}$.

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5. Given $\cos(x+y) = y \sin x$, find $\dfrac{dy}{dx}$.

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6. Given $\dfrac{3}{x} - \dfrac{2}{y} = 2x$, find $\dfrac{dy}{dx}$.

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7. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3,\ t \geq 0$. Find the velocity and acceleration functions.

8. Determine when the particle in problem #7 is moving up and when it is moving down.

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9. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$. Find (a) the rate of change of the volume with respect to the height if the radius is a constant and (b) the rate of change of the volume with respect to the radius if the height is constant.

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10. The volume of a cube in increasing at a rate of 10 cubic centimeters per minute. How fast is the surface area increasing when the length of an edge is 30 centimeters?

11. A paper cup has the shape of a cone with height 10 centimeters and radius 3 centimeters at the top. If water is poured into the cup at a rate of 2 cubic centimeters per second, how fast is the water level rising when the water is 5 centimeters deep?

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12. A balloon is rising at a constant speed of 5 feet per second. A boy is cycling along a straight road at a speed of 15 feet per second. When he passes under the balloon it is 45 feet above him. How fast is the distance between the boy and the balloon increasing 3 seconds later?

13. Find the linearization of $f(x) = \sqrt[3]{1+3x}$ at $x = 0$. Use the linearization to approximate the value of $\sqrt[3]{1.03}$.

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14. Find the linearization of $f(x) = \sqrt{25 - x^2}$ at $x = 3$.