Advanced Placement Calculus

Inverse Functions

Inverse Functions

1. Given the graph of a function, describe how you would determine if the function was one-to-one.

2. Use the derivative to show that $f(x) = 7x - 3$ has an inverse.

3. Use the derivative to show that $g(x) = \sqrt{x}$ has an inverse.

4. Use the derivative to show that $h(x) = x^4 + 5$ has an inverse.

5. Use the derivative to show that $f(x) = \dfrac{1+3x}{5-2x}$ has an inverse.

---

6. Use the derivative to show that $f(x) = \sqrt{2+5x}$ has an inverse.

Problems 7-9. ‘Assume the following functions are one-to-one. Find the value of $\left(f^{-1}\right)'(a)$ in two ways. First, actually find the inverse, take the derivative and evaluate it at a. Second, use the theorem which allows us to find the value of the derivative of an inverse without actually finding the inverse.

7. $f(x) = 2x + 1, \quad a = 3$

8. $f(x) = x^3, \;\; a = 8$

---

9. $f(x) = 9 - x^2, \;\; 0 \le x \le 3, \;\; a = 8$

Problems 10-13. Find $\left(f^{-1}\right)'(a)$ without actually finding the inverse.

10. $f(x) = x^3 + x + 1, \;\; a = 1$

11. $f(x) = x^5 - x^3 + 2x, \;\; a = 2$

---

12. $f(x) = \sqrt{x^3 + x^2 + x + 1}, \;\; a = 2$

---

13. $f(x) = \dfrac{1 + 3x}{5 - 2x}, \;\; a = 2$

1. In the space below, sketch the graphs of the following functions on the same coordinate grid:
   $f(x) = 2^x,\ f(x) = e^x,\ f(x) = 5^x,\ f(x) = 20^x$

---

2. In the space below, sketch the graphs of the following functions on the same coordinate grid:
   $f(x) = 3^x,\ f(x) = e^{-x},\ f(x) = 8^x,\ f(x) = 8^{-x}$

---

3. In the space below, sketch the graphs of the following functions on the same coordinate grid:
   $f(x) = 0.9^x,\ f(x) = 0.6^x,\ f(x) = 0.3^x,\ f(x) = 0.1^x$

4. Sketch the graph of $y = 2^x$ and $y = 2^x + 1$ on the same set of axes without your calculator.

---

5. Sketch the graph of $y = 3^x$ and $y = 3^{-x}$ on the same set of axes without your calculator.

---

6. Sketch the graph of $y = 3^x$ and $y = -3^{-x}$ on the same set of axes without your calculator.

---

7. Evaluate: $\lim_{x \to \infty} (1.1)^x$

8. Evaluate: $\lim_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

---

9. Evaluate: $\lim_{x \to -\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

---

10. Evaluate: $\lim_{x \to \pi/2} e^{\dfrac{2}{x-1}}$

---

11. Given $f(x) = e^{\sqrt{x}}$, find $f'(x)$.

12. Given $y = xe^{2x}$, find $\dfrac{dy}{dx}$.

13. Given $h(t) = \sqrt{1 - e^t}$, find $h'(t)$.

14. Given $y = e^{x \cos x}$, find $\dfrac{dy}{dx}$.

15. Given $g(x) = e^{-1/x}$, find $g'(x)$.

16. Given $y = \tan\left(e^{3x-2}\right)$, find $\dfrac{dy}{dx}$.

---

17. Given $f(x) = \dfrac{e^{3x}}{1+e^{x}}$, find $f'(x)$.

---

18. Given $y = x^{e}$, find $\dfrac{dy}{dx}$.

---

19. Given $y = \dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$, find $\dfrac{dy}{dx}$.

20. Given $f(x) = \sec\left(e^{\tan x^2}\right)$, find $f'(x)$.

---

21. Write an equation of a tangent to the curve $y = e^{-x} \sin x$ at the point $(\pi, 0)$.

---

22. Write an equation of a tangent to the curve $f(x) = x^2 e^{-x}$ at the point $\left(1, \dfrac{1}{e}\right)$.

23. Given $\cos(x-y) = xe^x$, find $\dfrac{dy}{dx}$.

---

24. Find an equation of the tangent to the curve $2e^{xy} = x + y$ at the point $(0, 2)$.

1. Express as a single logarithm: $\log_5 a + \log_5 b - \log_5 c$

2. Express as a single logarithm: $\log_2 x + 5\log_2(x+1) + \dfrac{1}{2}\log_2(x-1)$

3. Express as a single logarithm: $2\ln 4 - \ln 2$

4. Express as a single logarithm: $\dfrac{1}{3}\ln x - 4\ln(2x+3)$

5. On the same set of axes, sketch the graphs $y = \ln x$ and $y = -\ln x$.

6. On the same set of axes, sketch the graphs $y = \ln x$ and $y = -\ln(-x)$.

7. On the same set of axes, sketch the graphs $y = \ln x$ and $y = \ln(x^2)$.

8. On the same set of axes, sketch the graphs $y = \ln x$ and $y = \ln(x + 3)$.

9. On the same set of axes, sketch the graphs $y = \ln x$ and $y = 2 + \ln x$.

Problems 10-19: Solve for x.

10. $\log_2 x = 3$

11. $e^x = 16$

12. $\ln(2x - 1) = 3$

13. $3^{x+2} = m$

14. $\ln x = \ln 5 + \ln 8$

15. Solve for x: $\ln\left(e^{2x-1}\right) = 5$

16. $\ln x + \ln(x-1) = 1$

17. $\ln(\ln x) = 1$

18. $2^{3^x} = 5$

19. $\ln(x+6) + \ln(x-3) = \ln 5 + \ln 2$

Derivatives of Logarithmic Functions

Problems 1-3: Find $f'(x)$.

1. $f(x) = \ln(x+1)$

2. $f(x) = x^2 \ln(1-x^2)$

3. $f(x) = \log_3(x^2-4)$

4. Given $f(x) = x \ln x$, find $f'(x)$ and $f''(x)$.

5. Given $g(x) = \log_{10} x$, find $g'(x)$ and $g''(x)$.

6. Given $f(x) = \sqrt{x} \ln x$, find $f'(x)$.

7. Given $g(x) = \ln \dfrac{a-x}{a+x}$ where a is a constant, find $g'(x)$.

8. Given $g(x) = \ln \sqrt{x}$, find $g'(x)$.

9. Given $f(t) = \log_2(t^4 - t^2 + 1)$, find $f'(t)$.

10. Given $g(u) = \dfrac{1 - \ln u}{1 + \ln u}$, find $g'(u)$.

11. Given $y = [\ln(\sin x)]^3$, find $\dfrac{dy}{dx}$.

12. Given $y = \dfrac{\ln x}{1 + x^2}$, find $\dfrac{dy}{dx}$.

13. Given $f(x) = e^x \ln x$, find $f'(x)$.

14. Given $f(t) = \pi^{-t}$, find $f'(t)$.

15. Given $h(t) = t^3 - 3^t$, find $h'(t)$.

16. Given $y = \ln\left(e^{-x} + xe^{-x}\right)$, find $\dfrac{dy}{dx}$.

17. Given $y = x^{\sin x}$, find $\dfrac{dy}{dx}$.

18. Given $y = x^{e^x}$, find $\dfrac{dy}{dx}$.

19. Write an equation of the tangent line to the curve $y = \ln(\ln x)$ at the point $(e, 0)$.

20. Write an equation of the tangent line to the curve $y = 10^x$ at the point $(1, 10)$.

21. Given $f(x) = 2x + \ln x$, find $\left(f^{-1}\right)'(2)$.

1. A common inhabitant of human intestines is the bacterium escherichia coli. A cell of this bacterium in a nutrient broth medium divides into two cells every 20 minutes. The initial population of a culture is 100 cells. Find (a) an expression for the number of cells after t hours and (b) the number of cells present after 10 hours and (c) the time when the population will reach 10,000 cells.

2. A bacteria culture starts with 4000 bacteria and the population triples every half-hour. Find (a) an expression for the number of bacteria after t hours and (b) the number of bacteria present after 20 minutes and (c) the time when the population will reach 20,000.

3. A bacteria culture starts with 500 bacteria and after 3 hours there are 8000 bacteria. Find (a) an expression for the number of bacteria after t hours and (b) the number of bacteria present after 4 hours and (c) the time when the population will reach 30,000.

4. The count in a bacteria culture was 400 after 2 hours and 25,600 after 6 hours. Find (a) the initial population of the culture (b) an expression for the number of bacteria after t hours (c) how long it will take the population to double and (d) the time when the population reaches 100,000.

---

5. The table below gives estimates of the world population, in millions,over two centuries.

Year	1750	1800	1850	1900	1950
Population	728	906	1171	1608	2517

   (a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.

   (b) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 1992. Compare with the actual population of 5.4 billion and try to explain the discrepancy.

6. Suppose \$20,000 is invested in an account earning 8% continuous interest. Find a function which will yield the total amount of money in the account after t years. Use this function to determine when the investment will be worth \$30,000 and find out how long it will take for the original investment to double.

7. A bank advertises that it compounds interest continuously and that it will double your money in 12 years. What is the annual interest rate?

8. The rate of growth in the demand for coal in the world in 4%. When will the demand be double of that in 1996?

9. In 1970 the cost of a double-dip ice cream cone was \$.52. In 1978 it was \$.66. Assuming the exponential model, (a) find a function which will yield the cost of a cone at any time t, (b) find the cost of a cone in 1998 and (c) determine in what year the cost doubled from its original.

10. Peter Minuit of the Dutch West India Company purchased Manhattan Island from the Indians in 1626 for \$24 worth of merchandise. Assuming an exponential model and rate of inflation of 5%, how much was Manhattan worth in 1999?

11. You would like to make a one-time investment of \$1000 in a stock. You would like the value of the stock to reach \$100,000 in 50 years. Assuming the exponential growth model, what rate would the stock have to pay?

12. The cost of a first class postage stamp in 1962 was \$.04. In 1995 it was \$.32. Find (a) the growth rate and (b) estimate the cost of the stamp in 1999.

Inverse Trigonometric Functions

Problems 1-8: Find the exact value of each of the following expressions. Use reference angles and the unit circle.

1. $\cos^{-1}(-1)$

2. $\tan^{-1}\left(\sqrt{3}\right)$

3. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

4. $\sin\left(\sin^{-1}(0.7)\right)$

5. $\sin\left(\cos^{-1}\frac{4}{5}\right)$

---

6. $\arcsin\left(\sin\frac{5\pi}{4}\right)$

---

7. $\sin^{-1}(1)$

---

8. $\cos^{-1}(0)$

9. Given $f(x) = \sin^{-1}(2x - 1)$, find $f'(x)$.

---

10. Given $y = \left(\sin^{-1} x\right)^2$, find $\dfrac{dy}{dx}$.

---

11. Given $f(x) = \left(\sin^{-1} x\right)(\ln x)$, find $f'(x)$.

12. Given $f(t) = \dfrac{\cos^{-1} t}{t}$, find $f'(t)$.

13. Given $F(t) = \sqrt{1 - t^2} + \sin^{-1} t$, find $F'(t)$.

14. Given $y = \sin^{-1}(x^2)$, find $\dfrac{dy}{dx}$.

15. Given $y = \sec^{-1}\sqrt{1+x^2}$, find $\dfrac{dy}{dx}$.

16. Given $y = \tan^{-1}(\sin x)$, find $\dfrac{dy}{dx}$.

17. Given $f(x) = \left(\tan^{-1} x\right)^{-1}$, find $f'(x)$.

18. Given $y = x^2 \cot^{-1}(3x)$, find $\dfrac{dy}{dx}$.

---

19. A ladder 10 feet long leans against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a speed of 2 feet per second, how fast is the angle between the top of the ladder and the wall changing when the bottom of the ladder is 6 feet from the base of the wall?

Inverse Functions Review

Problems 1-5: Determine if the following statements are true or false. Include a justification for your answer.

1. The function $f(x) = \cos x$ where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, is one-to-one.

2. You can always divide by e^x.

3. $\frac{d}{dx} \ln 10 = \frac{1}{10}$

4. $\cos^{-1} x = \frac{1}{\cos x}$

5. $\dfrac{d}{dx}\log_8 x = \dfrac{1}{(3\ln 2)x}$

Problems 6-9: Solve for x.

6. $e^x = 5$

---

7. $\log_{10} e^x = 1$

---

8. $\ln\left(x^\pi\right) = 2$

---

9. $\tan x = 4$

10. Given $y = \log_{10}(x^2 - x)$, find $\dfrac{dy}{dx}$.

---

11. Given $y = \left(\sqrt{2}\right)^x$, find $\dfrac{dy}{dx}$.

---

12. Given $f(x) = e^{cx}(c\sin x - \cos x)$, where c is a constant, find $f'(x)$.

13. Given $g(x) = \ln\left(\sec^2 x\right)$, find $g'(x)$.

---

14. Given $y = x\, e^{1/x}$, find $\dfrac{dy}{dx}$.

---

15. Given $f(x) = 7^{\sqrt{2x}}$, find $f'(x)$.

16. Given $y = e^{e^x}$, find $\dfrac{dy}{dx}$.

17. Given $f(x) = \ln \dfrac{1}{x} + \dfrac{1}{\ln x}$, find $f'(x)$.

18. Given $y = \sin^{-1}\left(\dfrac{x-1}{x+1}\right)$, find $\dfrac{dy}{dx}$.

19. Find an equation of the tangent to the curve $y = \ln\left(e^x + e^{2x}\right)$ at the point $(0, \ln 2)$.

---

20. Find an equation of the tangent to the curve $y = x\ \ln x$ at the point (e, e).

---

21. At what point on the curve $y = [\ln(x + 4)]^2$ is the tangent horizontal?

22. Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.

23. Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

24. A bacteria culture starts with 1000 bacteria and the growth rate is proportional to the number of bacteria. After 2 hours the population is 9000. Find the number of bacteria after 3 hours. In what period of time will the population double?

25. Given $f(x) = g(e^x)$, find $f'(x)$ in terms of $g'(x)$.

---

26. Given $f(x) = e^{g(x)}$, find $f'(x)$ in terms of $g'(x)$.

---

27. Given $f(x) = g(\ln x)$, find $f'(x)$ in terms of $g'(x)$.

---

28. Given $f(x) = x\, e^{g(\sqrt{x})}$, find $f'(x)$ in terms of $g'(x)$.

---

29. Given $f(x) = \ln\left[g\left(e^x\right)\right]$, find $f'(x)$ in terms of $g'(x)$.

30. Use the derivative to show that $f(x) = \dfrac{x+3}{3x-1}$ has an inverse.

---

31. Use the derivative to show that $f(x) = \sqrt{x-2}$ has an inverse.

---

32. Use the derivative to show that $f(x) = x^5 + x^3 + x$ has an inverse.

---

33. Find the inverse of $f(x) = \dfrac{x+3}{x-6}$.

---

34. Find the inverse of $f(x) = e^{3x-4}$.