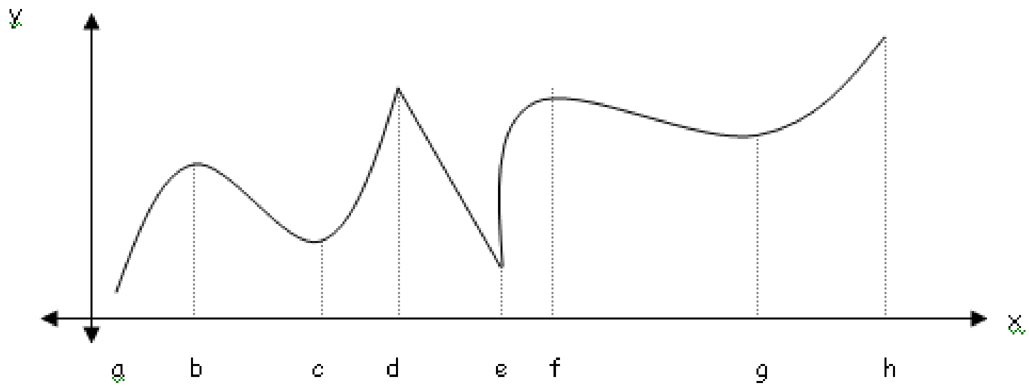
Advanced Placement Calculus

Applications of the Derivative II

Maximum and Minimum Function Values

1. State whether the function whose graph is shown below has an absolute or relative (local) minimum or maximum at the specified x values. Assume the endpoints are included.



Problems 2-9. Sketch each of the following curves. Using the sketch and function values, find the relative (local) and absolute extrema of each function. Not all functions will have extrema.

2. $f(x) = 1 + 2x,\ x \geq -1$

3. $f(x) = 1 - x^2,\ 0 < x < 1$

4. $f(x) = 1 - x^2,\ 0 \leq x \leq 1$

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5. $f(x) = 1 - x^2,\ -2 \leq x \leq 1$

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6. $f(t) = \dfrac{1}{t},\ 0 < t < 1$

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7. $f(x) = \sin x,\ -2\pi \leq x \leq 2\pi$

8. $f(x) = x^5$

9. $f(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1 \\ 2 - x & \text{if } 1 \leq x \leq 2 \end{cases}$

Find the critical numbers of each of the following functions.

10. $f(x) = 2x - 3x^2$

11. $f(x) = x^3 - 3x + 1$

12. $f(t) = 2t^3 + 3t^2 + 6t + 4$

13. $s(t) = 2t^3 + 3t^2 - 6t + 4$

14. $g(x) = \sqrt[9]{x}$

15. $g(t) = 5t^{2/3} + t^{5/3}$

16. $f(r) = \dfrac{r}{r^2+1}$

17. $f(x) = x^{4/5}(x-4)^2$

18. $f(x) = \sin 2x$ on $[0, 2\pi]$

19. $f(x) = x \ln x$

Problems 20-28: Find the absolute extrema of the following functions on the specified closed interval.

20. $f(x) = x^2 - 2x + 2$ on $[0, 3]$

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21. $f(x) = x^3 - 12x + 1$ on $[-3, 5]$

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22. $f(x) = 2x^3 + 3x^2 + 4$ on $[-2, 1]$

23. $f(x) = x^4 - 4x^2 + 2$ on $[-3, 2]$

24. $f(x) = x^2 + \dfrac{2}{x}$ on $\left[\dfrac{1}{2}, 2\right]$

25. $f(x) = x^{4/5}$ on $[-32, 1]$

26. $f(x) = xe^{-x}$ on $[0, 1]$

27. $f(x) = \dfrac{\ln x}{x}$ on $[1, 3]$

28. Sketch a graph of a function on $[0, 1]$ that is discontinuous and has both an absolute maximum and an absolute minimum.

The Mean Value Theorem

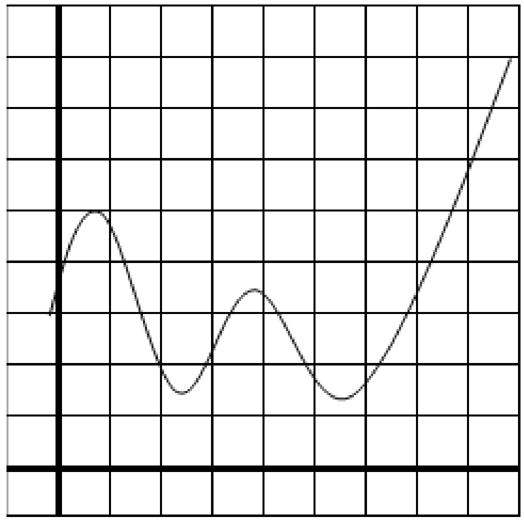
Problems 1-2: Verify that Rolle's Theorem holds for the given function on the given interval, then find the value of c that satisfies the conclusion of the theorem.

1. $f(x) = x^3 - x$ on $[-1, 1]$

2. $f(x) = \cos 2x$ on $[0, 2\pi]$

3. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

4. Use the graph of f below to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[0, 8]$. (Hint–Use a straightedge and look for points on the curve where a tangent will be parallel to the secant line through the endpoints.)



Problems 5-7: Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all the numbers c that satisfy the conclusion of the Mean Value Theorem.

5. $f(x) = 1 - x^2$ on $[0, 3]$

6. $f(x) = \dfrac{1}{x}$ on $[1, 2]$

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7. $f(x) = 1 + \sqrt[3]{x-1}$ on $[2, 9]$

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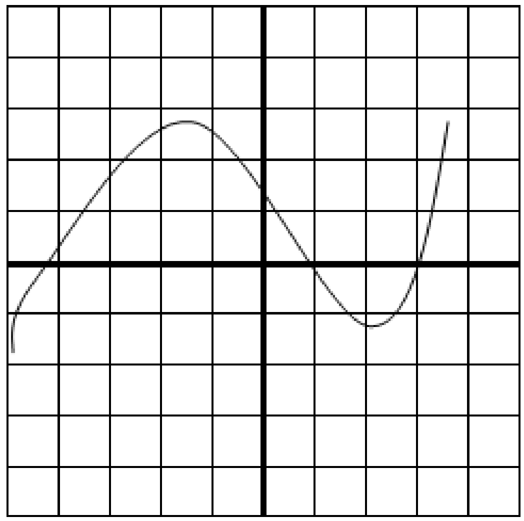
8. Let $f(x) = |x - 1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

1. The graph of the derivative of f is shown below. Determine the intervals on which f is increasing, the intervals where f is decreasing, the values of x where f has a relative maximum and the values of x where f has a relative minimum.



Problems 2-8: Find the intervals on which the following are increasing or decreasing. then find the relative extrema (if any). Make sure you show all your mathematics. Verify your answers with your calculator.

2. $f(x) = 20 - x - x^2$

3. $f(x) = x^3 + x + 1$

4. $f(x) = 2x^2 - x^4$

5. $f(x) = x^3(x-4)^4$

6. $f(x) = x^{1/5}(x+1)$

7. $f(x) = x\sqrt{x - x^2}$

8. $f(x) = x - 2\sin x$ on $[0, 2\pi]$

Problems 9-12: Find the intervals on which the following functions are increasing or decreasing. You do not need to find any relative extrema.

9. $f(x) = x^3 + 2x^2 - x + 1$

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10. $f(x) = x^6 + 192x + 17$

11. $f(x) = xe^x$

12. $f(x) = \dfrac{\ln x}{\sqrt{x}}$

13. Find the relative and absolute extrema (if any) of: $f(x) = x + \sqrt{1-x}$ on $[0, 1]$. Justify your answers.

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14. Find the relative and absolute extrema (if any) of: $g(x) = \dfrac{x}{x^2+1}$ on $[-5, 5]$.Justify your answers.

15. Sketch the graph of a function that meets the following conditions:
$f(1) = 5,\ f(4) = 2,\ f'(1) = 0,\ f'(4) = 0,\ f'(x) > 0$ for $x < 1,\ f'(x) \leq 0$ for $x > 1$.

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16. Sketch the graph of a function that meets the following conditions:
$f'(5) = 0,\ \lim_{x \to 3} f(x) = -\infty,\ f'(x) < 0$ for $x < 3$ and $x > 5,\ f'(x) > 0$ for $3 < x < 5$.

Concavity and Points of Inflection

1. Use the Second Derivative Test to find the relative extrema of $f(x) = x^3 - x$.

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2. Use the Second Derivative Test to find the relative extrema of $f(x) = 2x^3 + 5x^2 - 4x$.

Problems 3-12: Determine where the functions are concave up or concave down. Find all (if any) inflection points.

3. $f(x) = x^3 - x$

4. $f(x) = x^4 - 6x^2$

5. $f(x) = 3x^5 - 5x^3 + 3$

6. $P(x) = x\sqrt{x^2 + 1}$

7. $f(x) = x^{1/3}(x+3)^{2/3}$

8. $h(\theta) = \sin^2 \theta$ on $[0, 2\pi]$

9. $y = 6x^2 - 2x^3 - x^4$

10. $y = \dfrac{x}{(1+x)^2}$

11. $y = xe^x$

12. $f(x) = \dfrac{\ln x}{\sqrt{x}}$

13. Sketch the graph of a function that satisfies all of the following properties:
$f'(x) > 0$ and $f''(x) < 0$ for all x.

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14. Sketch the graph of a function that satisfies all of the following properties:
$\lim_{x \to 3} f(x) = -\infty,\ f''(x) < 0$ if $x \neq 3,\ f'(0) = 0,\ f'(x) > 0$ when $x < 0$ or $x > 3,\ f'(x) < 0$ when $0 < x < 3$.

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15. Sketch the graph of a function that satisfies all of the following properties:
$f'(-1) = f'(1) = 0,\ f'(x) < 0$ when $|x| < 1,\ f'(x) > 0$ when $|x| > 1,\ f(-1) = 4,\ f(1) = 0,\ f''(x) < 0$ when $x < 0,\ f''(x) > 0$ when $x > 0$.

16. Sketch the graph of a function that satisfies all of the following properties:
$f'(-1) = 0,\ f'(2) = 0,\ f(-1) = f(2) = -1,\ f(-3) = 4,\ f'(x) = 0$ when $x < -3$,
$f'(x) < 0$ on $(-3, -1)$ and $(0, 2)$, $f'(x) > 0$ on $(-1, 0) \cup (2, \infty)$, $f''(x) > 0$ on $(-3, 0) \cup (0, 5)$,
$f''(x) < 0$ on $(5, \infty)$.

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17. Sketch the graph of a function that satisfies all of the following properties:
$f(0) = 0,\ f(-1) = 1,\ f'(-1) = 0,\ f''(x) > 0$ on $(-\infty, -1)$, $f''(x) < 0$ on $(-1, 0) \cup (0, \infty)$, $f'(x) > 0$ for $x > 0$.

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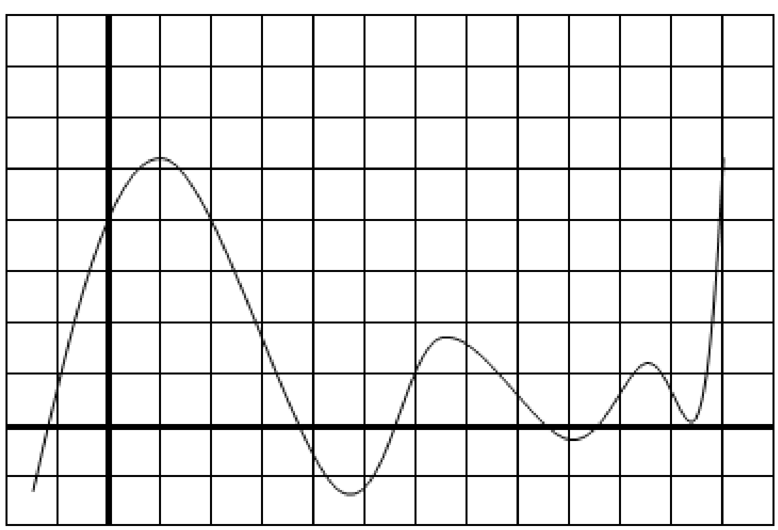
18. The graph of f' is given below. Determine the intervals on which f is increasing or decreasing, concave up or down. Also find the x-coordinates where f has relative extrema and inflection points.

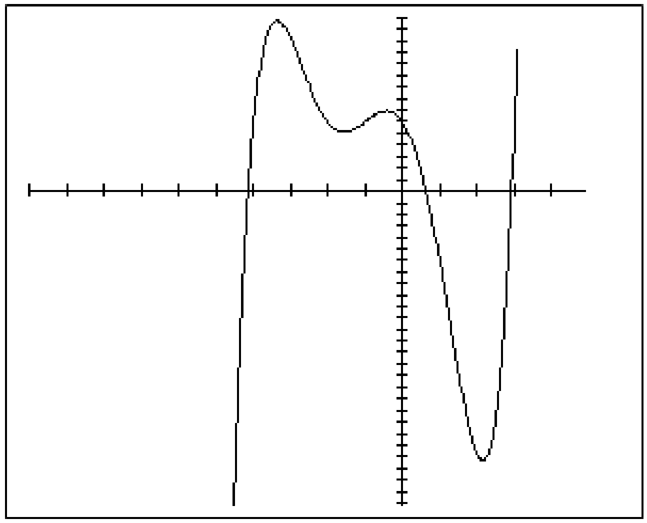


Analysis of a Function

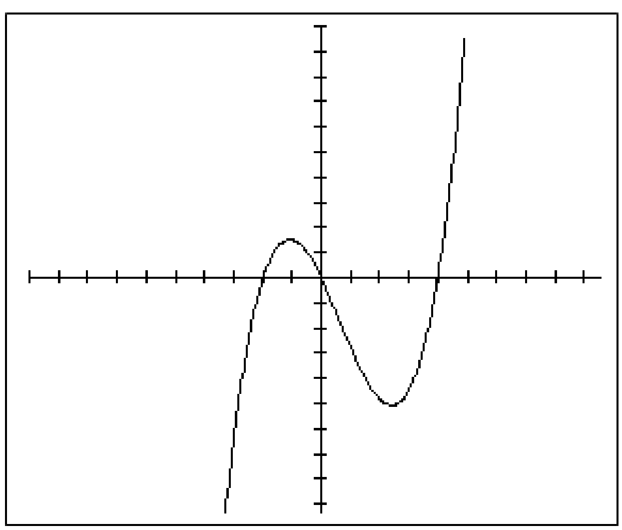
For each of the following functions determine where it is increasing/decreasing, relative extrema, where it is concave up/down, points of inflection, vertical, horizontal and oblique asymptotes.

1. $f(x) = 1 - 3x + 5x^2 - x^3$
2. $f(x) = x^4 - 6x^2$
3. $f(x) = \dfrac{x}{x-1}$
4. $y = \dfrac{1}{x^2 - 9}$
5. $y = \dfrac{1}{(x-1)(x+2)}$
6. $g(x) = \dfrac{1+x^2}{1-x^2}$
7. $h(x) = \dfrac{1}{x^3 - x}$
8. $y = x\sqrt{x+3}$
9. $y = \sqrt{x^2+1} - x$
10. $f(x) = \dfrac{\sqrt{1-x^2}}{x}$
11. $y = x \tan x$ on $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
12. $f(x) = \sin 2x - 2\sin x$ on $(0, 2\pi)$
13. $g(x) = e^{\frac{-1}{e^{x+1}}}$
14. $y = \dfrac{1}{1+e^{-x}}$
15. $y = \ln\left(1+x^2\right)$
16. $h(x) = x \ln x$
17. $y = \dfrac{e^x}{x}$
18. $f(x) = \dfrac{x^3}{x^2-1}$
19. $f(x) = \dfrac{x^2}{2x+5}$
20. $f(x) = \dfrac{x}{2} - \sin x$ on $(0, 3\pi)$

1. Given the following graph of f, sketch the graph of f'. Draw directly on top of the graph of f.



2. Given the following graph of f, sketch the graph of f'. Draw directly on top of the graph of f.



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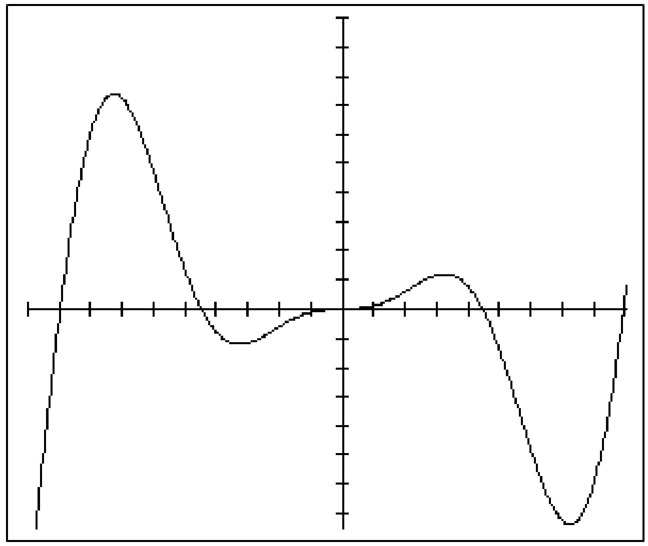

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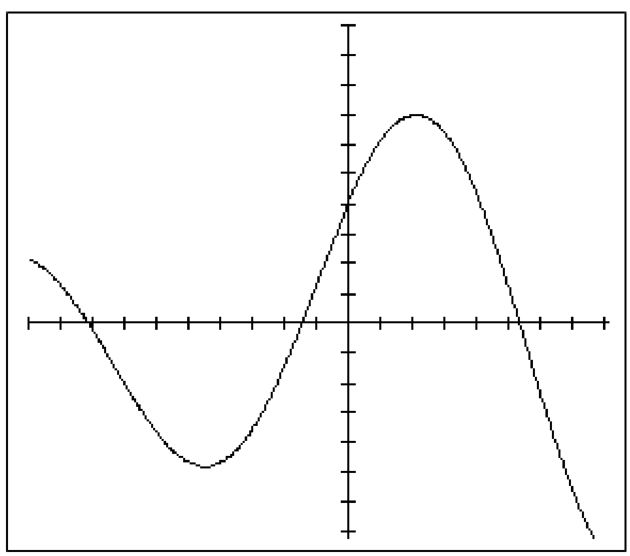

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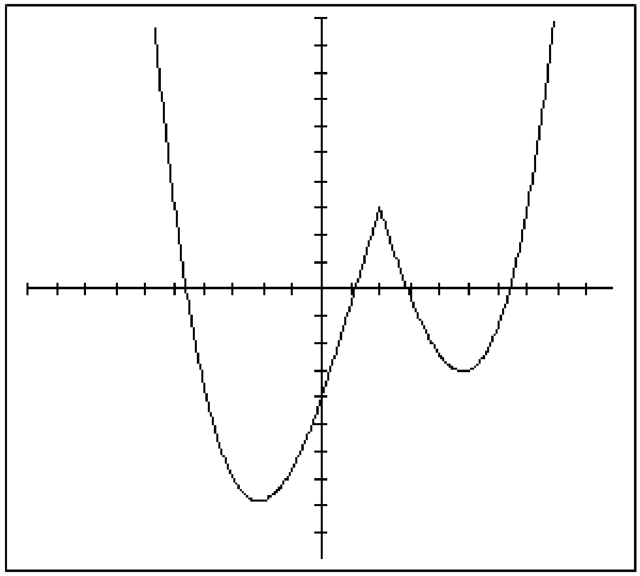

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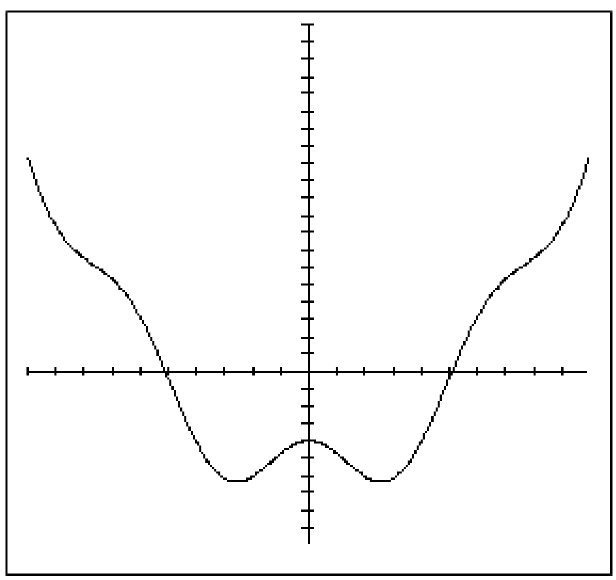

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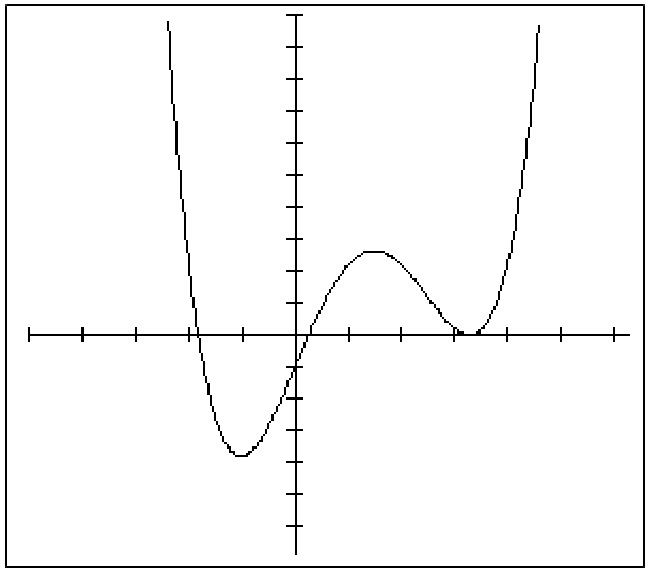

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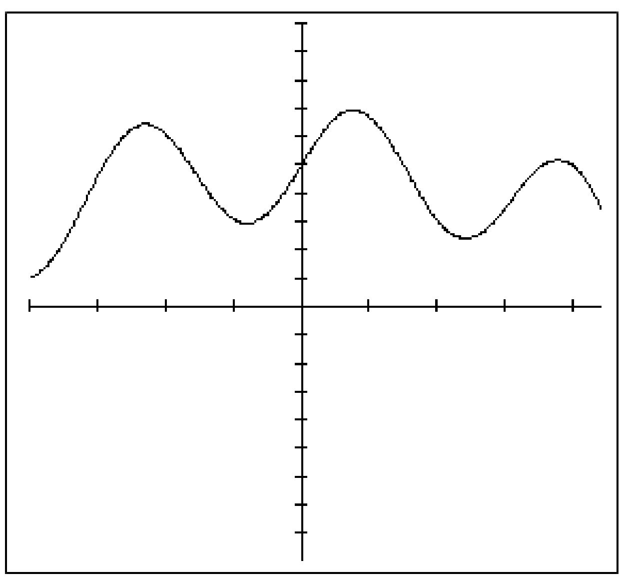

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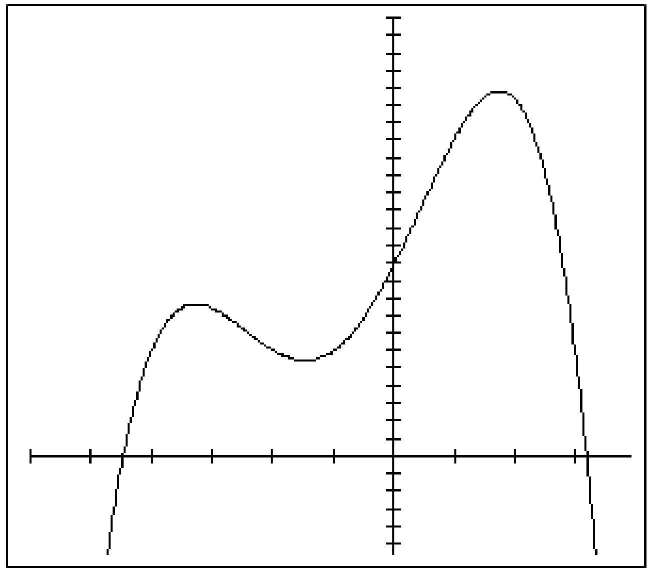

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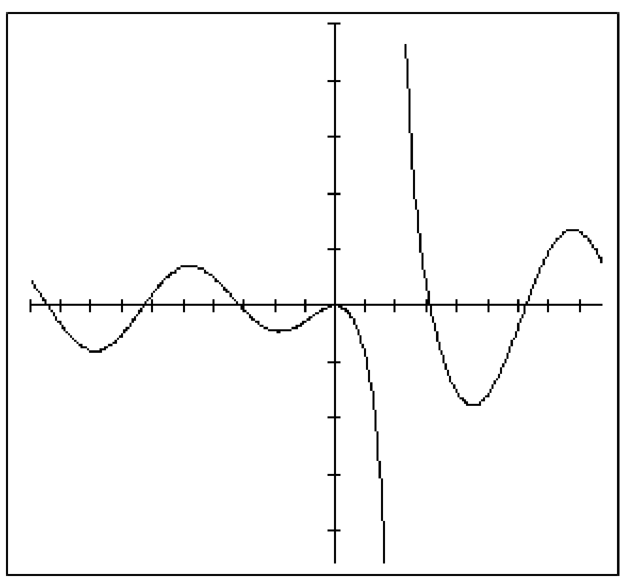

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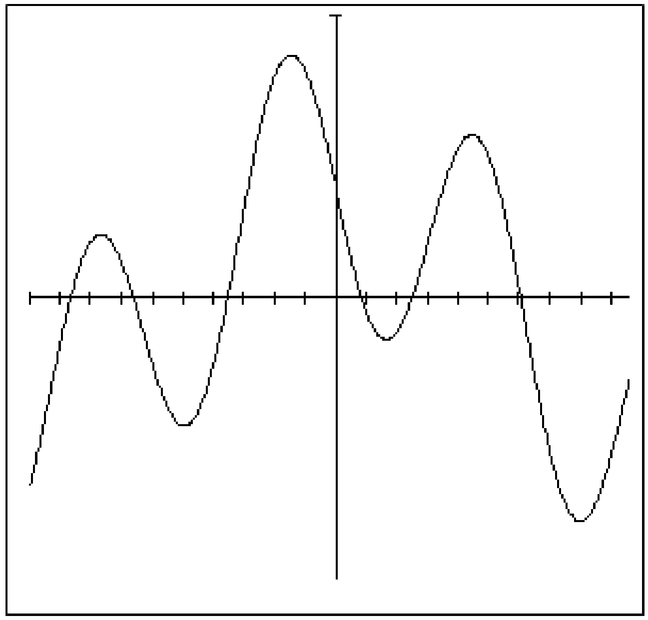

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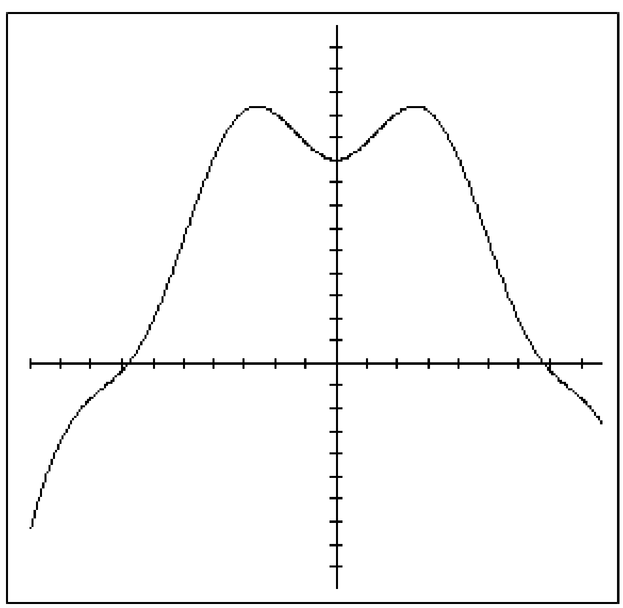

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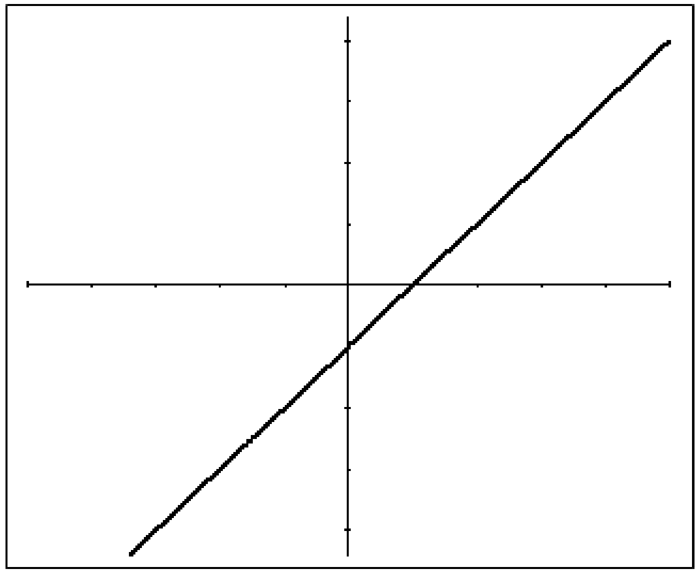

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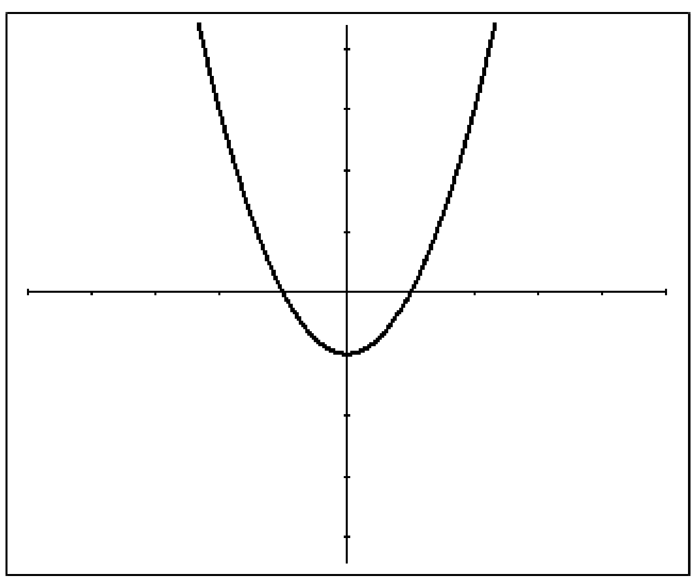

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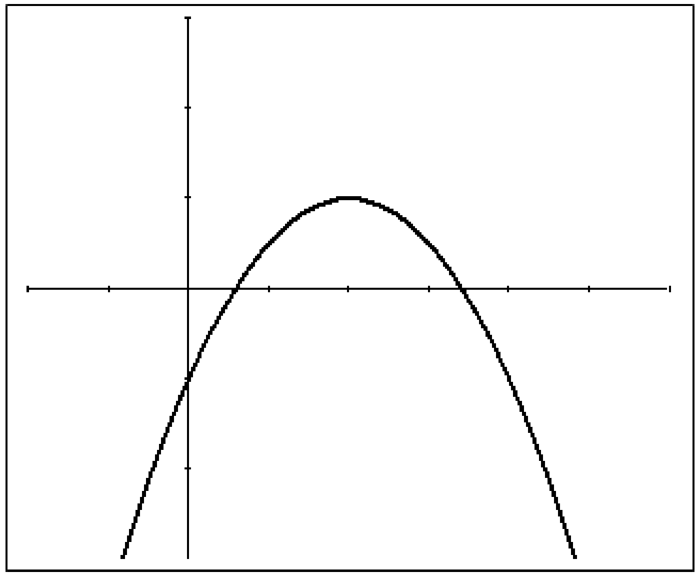

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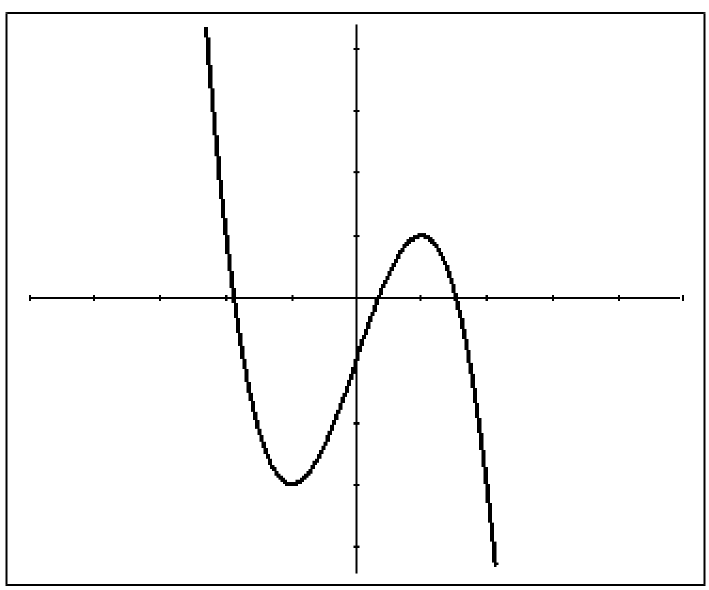

1. Find two numbers whose sum is 100 and whose product is a maximum.

2. Find two numbers whose product is 100 and whose sum is a minimum.

3. Show that of all the rectangles with a given perimeter, the one with the greatest area is a square.

4. A farmer with 750 feet of fencing want to enclose a rectangular area and then divide it into four pens with fencing parallel to the short side of the rectangle. What is the largest possible total area of the four pens?

5. If 1200 square meters of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

6. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of the materials for the cheapest such container.

7. A box with an open top is to be constructed from a square piece of cardboard, 3 feet wide by cutting a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

8. Find the point on the line $y = 2x - 3$ that is closest to the origin.

9. Find the points on the hyperbola $y^2 - x^2 = 4$ that are closest to the point $(2, 0)$.

10. Find the dimensions of the rectangle with the largest area that can be incribed in a circle of radius r.

11. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

12. A painting in an art gallery has height 7 feet. Its lower edge is to be placed 2 feet above eye level. How far from the painting should an observer stand to maximize the viewing angle?

13. A fence 8 feet tall runs parallel to a tall building at a distance of 4 feet from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?