Advanced Placement Calculus

Additional Definite Integral Topics

Average Value of a Function
Definite Integral as an Accumulator

Average Value

Problems 1-3: Use the property $m(b-a) \leq \int\limits_a^b f(x)\, dx \leq M(b-a)$ to find a lower and upper bound on the following integrals...without integrating!

1. $\int\limits_1^2 \frac{1}{x}\, dx$

2. $\int\limits_0^2 \sqrt{x^3+1}\, dx$

3. $\int\limits_{-1}^1 \sqrt{1+x^4}\, dx$

Problems 4-7: Find the average value of the following functions on the given interval.

4. $f(x) = x^2 - 2x$ on $[0, 3]$.

5. $f(x) = x^4$ on $[-1, 1]$.

6. $f(x) = \sin^2 x \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{4}\right]$.

7. $f(x) = \frac{1}{x}$ on $[1, 4]$.

Problems 8-11: Find the average value of the given function on the given interval and then find the value of c such that $f_{avg} = f(c)$.

8. $f(x) = 4 - x^2$ on $[0, 2]$.

9. $f(x) = 4x - x^2$ on $[0, 3]$.

10. $f(x) = x^3 - x + 1$ on $[0, 2]$.

11. $f(x) = x \sin x^2$ on $[0, \sqrt{\pi}]$.

1. Given that $h'(t)$ is the rate of change in a child's height measured in inches per year, what does the integral $\int_0^{10} h'(t)\ dt$ represent and what are its units?

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2. Given that $r'(t)$ is the rate of change in the radius of a spherical balloon measured in centimeters per second, what does the integral $\int_1^2 r'(t)\ dt$ represent and what are its units?

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3. Given that $v(t)$ is the velocity of a particle in rectilinear motion, measured in centimeters per hour, what does the integral $\int_{t_1}^{t_2} v(t)\ dt$ represent and what are its units?

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4. Suppose that sludge is emptied into a river at the rate of $V(t()$ gallons per minute, starting at time $t = 0$. Write an integral that represents the total volume of sludge that is emptied into the river during the first hour.

5. The marginal cost (the cost of the nth item) is given by $C'(x) = 2x + 1$. If the cost of producing 2 items is \$50, find (a) the total cost function, (b) the cost of making 50 items and (c) the cost of making the 9th through the 100th item.

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6. A particle's velocity is given by $v(t) = t^2 - 2t - 8$. Find the net distance and the total distance traveled from $t = 1$ to $t = 6$.

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7. A particle's velocity is given by $v(t) = .5 - te^{-t}$. Find the net distance and the total distance traveled from $t = 0$ to $t = 5$.

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8. A particle moves along the x-axis so that at any time t, $0 \leq t \leq 5$, its velocity is given by $v(t) = \sin t + e^{-t}$. When $t = 0$, the particle is at the origin. (a) Write an expression for the position $x(t)$ of the particle at any time t, $0 \leq t \leq 5$. (b) Find all the values of t for which the particle is at rest. (c) For $0 \leq t \leq 5$, find the average value of the position function determined in part (a). (d) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

9. An animal population is increasing at a rate of $200 + 50t$ per year (where t is measured in years). By how much does the animal population increase between the fourth and tenth years?

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10. An engineer studying the power consumption of a manufacturing plant determines that the plant's daily rate of electricity usage in kilowatts per hour can be reasonably modeled by the formula: $R(t) = 2000e^{-t/48} + 500\sin\left(\frac{\pi}{12}t\right)$ where $0 \leq t \leq 24$. (a) How many kilowatts of electricity does the plant use in a 24-hour period? (b) Find the average rate of electricity usage over the first 8 hours. (c) Determine the maximum rate of electricity usage during the first 8-hour period to 3 decimal places.

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11. A particle moves along the x-axis with a velocity given by $v(t) = e^t - 2$. Find the total and net distance the particle travels from $t = 0$ to $t = 3$.

12. A particle moves along the x-axis with a velocity given by $v(t) = t^3 - 3t^2 + 2t$. Find the total and net distance traveled from $t = 0$ to $t = 3$.

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13. A particle moves along the x-axis with a velocity given by $v(t) = \sin t$. Find the total and net distance traveled from $t = 0$ to $t = \frac{\pi}{2}$.

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14. A particle moves along the x-axis with a velocity given by $v(t) = |t - 3|$. Find the total and net distance traveled from $t = 0$ to $t = 5$.

15. If the average American's annual income is changing at a rate given in dollars per month by $r(t) = 40(1.002)^t$ where t is in months from January 1,2000. What change in income can the average American expect during the year 2000?

16. A cup of coffee at 90 degrees centigrade is put into a 20 degree room when $t = 0$. If the coffee's temperature is changing at a rate given in degrees centigrade per minute by $r(t) = -7e^{-0.1t}$, t in minutes, estimate, to one decimal place, the coffee's temperature when $t = 10$.

17. The graph below shows the function $R(t)$ which describes the rate (in gallons per hour) that water is leaking out of a container, where t is measured in hours. Write an integral which would express the total amount of water that leaks our in the first 2 hours. Use the graph to estimate the total amount of water that leaks out in the first 5 hours and in the first 10 hours.