Summary Information

Some Useful Trigonometric Identities

$\sin x \csc x = 1$	$\cos x \sec x = 1$	$\tan x \cot x = 1$
$\sin^2 x + cos^2 x = 1$	$1 + \tan^2 x = \sec^2 x$	$1 + \cot^2 x = \csc^2 x$
$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\sin x = \tan x \cos x$
$\cos x = \cot x \sin x$	$\sin 2x = 2 \sin x \cos x$	$\cos 2x = \cos^2 x - \sin^2 x$
$\cos 2x = 2\cos^2 x - 1$	$\cos 2x = 1 - 2\sin^2 x$	$\sin(a+b) = \sin a \cos b + \cos a \sin b$
$\sin(a-b) = \sin a \cos b - \cos a \sin b$	$\cos(a+b) = \cos a \cos b - \sin a \sin b$	$\cos(a-b) = \cos a \cos b + \sin a \sin b$

Tests for Symmetry

- If replacing x with $-x$ results in an equivalent equation, the graph is symmetric with respect to the y-axis.
- If replacing y with $-y$ results in an equivalent equation, the graph is symmetric with respect to the x-axis.
- If replacing x with $-x$ and y with $-y$ results in an equivalent equation, the graph is symmetric with respect to the origin.

General Forms of Selected Standard Curves

- Circle: $(x-h)^2 + (y-k)^2 = r^2$
- Parabola: $y - k = a(x-h)^2$ Vertex at (h, k). If $a > 0$ opens up, if $a < 0$, opens down.
- Parabola: $a(y-k)^2 = x - h$ Vertex at (h, k). If $a > 0$ opens right, if $a < 0$, opens left.
- Cubic: $y - k = a(x-h)^3$ Standard cubic translated h units horizontally and k units vertically.

Odd and Even Functions

- If $f(x) = f(-x) \forall x$, f is even and is symmetric with respect to the y-axis.
- If $-f(x) = f(-x) \forall x$, f is odd and is symmetric with respect to the origin.

Periodicity

- $y = f(x)$ has period p if $f(x) = f(x+p) \forall x$.
- If $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$, the period is $\frac{2\pi}{|B|}$.

Limits

- Definition: $\lim_{x \to a} f(x) = L$ is true if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ such that whenever $|x - a| < \delta \to |f(x) - L| < \varepsilon$
- Limits as $x \to a$.
  - Always substitute a into the expression—one of 4 things will happen:
    - $\frac{0}{constant}$ yields 0.
    - $\frac{constant}{constant}$ yields a real number.
    - $\frac{constant}{0}$ means the limit does not exist and you must do left and right hand limits.

    * $\frac{0}{0}$ means you are not done and must use L'Hopital, rationalize or factor ... then start the process over.

- Limits as $x \to \infty$
  - If degree of numerator $>$ degree of denominator $\to$ limit does not exist and "answer" is $\pm \inf$
  - If degree of numerator $<$ degree of denominator $\to$ limit is zero
  - If degree of numerator $=$ degree of denominator $\to$ limit is ratio of coefficients of highest degree terms
- Trigonometric limits
  - $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$
  - $\lim_{x\to 0} \dfrac{1-\cos x}{x} = 0$

L'Hopital's Rule

If $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is of indeterminant form, and if both f and g have a limit as $x \to a$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

L'Hopital can be applied repeatedly.

Continuity

A function f is continuous at $x = a$ if $f(a) = \lim\limits_{x\to a} f(x)$.

This of course presupposes that the $\lim\limits_{x\to a} f(x)$ and $f(a)$ both exist.
If f is differentiable, f is continuous.
If f is continuous, f is not necessarily differentiable.
If f is not continuous, f is not differentiable.

The Derivative and Some Basic Derivative Theorems

Definition: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

Definition: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$$D_x \left[f(x)^n \right] = n f(x)^{n-1} f'(x)$$

$$D_x \left[f(x) g(x) \right] = f(x) g'(x) + g(x) f'(x)$$

$$D_x \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) f'(x) - f(x) g'(x)}{\left[g(x) \right]^2}$$

If $h(x) = f(g(x))$ then $h'(x) = f'(g(x)) g'(x)$

$$D_x[\sin f(x)] = f'(x) \cos f(x)$$

$$D_x[\cos f(x)] = -f'(x) \sin f(x)$$

$$D_x[\tan f(x)] = f'(x) \sec[f(x)]^2$$

$$D_x[\cot f(x)] = -f'(x) \csc[f(x)]^2$$

$$D_x[\sec f(x)] = f'(x) \sec[f(x)] \tan f(x)$$

$$D_x[\csc f(x)] = -f'(x) \csc f(x) \cot f(x)$$

$$D_x[\ln f(x)] = \frac{f'(x)}{f(x)}$$

$$D_x[\ln |f(x)|] = \frac{f'(x)}{f(x)}$$

$$D_x[e^{f(x)}] = f'(x) \cdot e^{f(x)}$$

$$D_x[a^{f(x)}] = f'(x) \cdot a^{f(x)} \cdot \ln a$$

$$D_x[\log_a f(x)] = \frac{f'(x)}{f(x) \cdot \ln a}$$

$$D_x[\log_a f(x)] = \frac{f'(x) \cdot \log_a e}{f(x)}$$

$$D_x[\sin^{-1} f(x)] = \frac{f'(x)}{\sqrt{1 - (f(x))^2}}$$

$$D_x[\cos^{-1} f(x)] = -\frac{f'(x)}{\sqrt{1 - (f(x))^2}}$$

$$D_x[\tan^{-1} f(x)] = \frac{f'(x)}{1 + (f(x))^2}$$

$$D_x[\cot^{-1} f(x)] = -\frac{f'(x)}{1 + (f(x))^2}$$

$$D_x[\sec^{-1} f(x)] = \frac{f'(x)}{f(x) \sqrt{(f(x))^2 - 1}}$$

$$D_x[\csc^{-1} f(x)] = -\frac{f'(x)}{f(x) \sqrt{(f(x))^2 - 1}}$$

Chain Rule: If $h(x) = f(g(x))$ then $h'(x) = f'(g(x)) g'(x)$.

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Rectilinear Motion

If $x(t)$ is a position function then $x'(t) = v(t)$ and $v'(t) = a(t)$ where v is the velocity function and a is the acceleration function.

Intermediate Value Theorem

If f is continuous on $[a, b]$ then for any k in $[f(a), f(b)]$ there exists a c in (a, b) such that $f(c) = k$.
This theorem basically guarantees us all the function values between any two other function values.
An important corollary states that if f is continuous on $[a, b]$ and $f(x)$ changes sign on the interval, then $f(x) = 0$ for some x in (a, b).

Rolle's Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b) = 0$, then there exists a c in (a, b) such that $f'(c) = 0$.

Mean Value Theorem for Integrals (Average Value of a Function on an Interval)

If f is continuous on $[a, b]$ and differentiable on (a, b) then there exists a z in (a, b) such that $f(z) = \frac{\int_a^b f(x)\,dx}{b-a}$.
$f(z)$ is called the "average value" of f on (a, b).

Extreme Value Theorem

If f is continuous on $[a, b]$ then f has an absolute maximum function value and an absolute minimum function value on $[a, b]$.
These absolute extrema may occur at endpoints or at critical numbers in (a, b).

Finding Absolute Extrema

- Find values critical numbers.
- Find function values at these critical numbers.
- Find function values at the endpoints of the interval.
- The largest of these values is the absolute maximum and the smallest is the absolute minimum.

Analysis of a Curve

If $f'(c) = 0$ or $f'(c)\nexists$ then $x = c$ is a "critical number".

The First Derivative Test for Relative Extrema:
If $f'(x) < 0$ before $x = c$ and $f'(x) > 0$ after $x = c$, then $f(c)$ is a relative minimum.
If $f'(x) > 0$ before $x = c$ and $f'(x) < 0$ after $x = c$, then $f(c)$ is a relative maximum.

The Second Derivative Test for Relative Extrema:
If $x = c$ is a critical number and $f''(c) > 0$ then $f(c)$ is a relative minimum.
If $x = c$ is a critical number and $f''(c) < 0$ then $f(c)$ is a relative maximum.

Concavity:
If $f''(x) > 0$ on (a, b), f is concave up on (a, b).
If $f''(x) < 0$ on (a, b), f is concave down on (a, b).

Values of x where $f''(x) = 0$ or $f''(x)\nexists$ represent possible inflection points.
$f''(x)$ must change signs across a possible inflection point in order for an inflection point to exist.

Horizontal and Vertical Asymptotes

If $f(c)\nexists$ and $\lim_{x \to c} f(x) = \pm\infty$, f has a vertical asymptote at $x = c$.

If $\lim_{x \to \pm\infty} f(x) = b$, then f has a horizontal asymptote at $y = b$.

Linearizations

If $y = f(x)$ is differentiable at $x = a$, then $L(x) = f(a) + f'(a)(x - a)$ is the linearization of f at $x = a$.
For values of x sufficiently close to $x = a$, $f(a) \approx L(a)$.

Linearizations are used to approximate *function values*.
A linearization is just a tangent line. If the curve is concave up, a tangent line approximation will underestimate a function value... $y(a) \leq f(a)$.
If the curve is concave down, a tangent line approximation will overestimate a function value... $y(a) \geq f(a)$.

Differentials

$$dy = f'(x)dx$$
$$dx = \Delta x$$
$$\Delta y = f(x + \Delta x) - f(x)$$
$dy \approx \Delta y$ for small dx's.

Differentials are used to approximate *changes in function values*.

Analysis of the Graphs of the Derivatives

Values of x where the graph of f' touches or crosses the x-axis are critical numbers of f.

If the graph of f' crosses from below to above the x-axis, f has a relative minimum.
If the graph of f' crosses from above to below the x-axis, f has a relative maximum.

If f' is increasing on an interval, $f'' > 0$ and f is concave up on the interval.
If f' is decreasing on an interval, $f'' < 0$ and f is concave down on the interval.

If the graph of f' has relative extrema at $x = a$, f has an inflection point at $x = a$.

The area bounded by the derivative and the x-axis is a measure of the about by which f increases or decreases.

Properties of the Definite Integral

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ where c is a constant.

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$$

If f is an odd function, then $\int_{-a}^a f(x)\,dx = 0$.

If f is an even function, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

The Fundamental Theorems of Calculus

First Fundamental Theorem:

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where } F'(x) = f(x).$$

Second Fundamental Theorem (little version):

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Second Fundamental Theorem (big version):

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\,h'(x) - f(g(x))\,g'(x).$$

Area

The area above the x-axis, below the curve $y = f(x)$ between $x = a$ and $x = b$ is given by:

$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(z)\,\Delta_i x = \int_a^b f(x)\,dx.$$

When finding area between a curve and the x-axis, you first need to know where the curve is above and below the axis. Areas below the axis must be subtracted to make them positive.

When finding area between two curves, all you need to do is think "top curve minus bottom curve" from a to b.

Numerical Integration

Trapezoidal Rule:

If the function f is continuous on $[a, b]$ and the numbers $x_0, x_1, x_2, \ldots$ form a partition on $[a, b]$ then:

$$\int_a^b f(x)\,dx = \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

Note: Most trapezoid questions on the AP test can be done by drawing a picture and simply adding the areas of the trapezoids. $\left(A = \frac{1}{2}(b_1 + b_2)h\right)$. *Rectangles*

The basic process is to partition the interval as instructed and then simply keep in mind that for a left sum, the rightmost x-value is not used and for a right sum, the leftmost x-value is not used.

Volumes of Revolution

Disk method:

$$V = \pi\int_a^b (radius)^2\,dx$$

Washer method:

$$V = \pi\int_a^b [(outer\,radius)^2 - (inner\,radius)^2]\,dx$$

Shell method:

$$V = 2\pi\int_a^b (radius)(height)\,dx$$

The Definite Integral as an Accumulator

If you integrate a rate of change, you get an amount of change.

If $g(x)$ describes the rate of change in a quantity, $\int_a^b g(x)\,dx$ yields the net amount of change if $g(x)$ changes sign on (a,b) and yields total amount of change if $g(x) > 0\,\forall x \in (a,b)$.

If $g(x)$ describes the rate of change in a quantity, $\int_a^b |g(x)|\,dx$ always yields the total amount of change.

If $v(t)$ describes the velocity, $\int_a^b v(t)\,dt$ yields the net distance traveled if $v(t)$ changes sign on (a,b) and yields total distance traveled if $v(t) > 0\,\forall x \in (a,b)$.

If $v(t)$ describes the velocity, $\int_a^b |v(t)|\,dt$ yields the total distance traveled.

To determine where an object is (as opposed to how far it went): $x(t_2) = x(t_1) + \int\limits_{t_1}^{t_2} v(t)\ dt$.

Inverse Functions

The functions f and g are inverses if and only if $f(g(x)) = x$ and $g(f(x)) = x$. (This is how you prove two functions are inverses.)

A function must be monotonic (one-to-one) in order to have an inverse.

A function f is monotonic $f'(x) \geq 0 \forall x$ or if $f'(x) \leq 0 \forall x$.

The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

If (c,d) is on f, then $(f^{-1})'(d) = \dfrac{1}{f'(c)}$.

The Inverse Trigonometric Functions

$y = \sin^{-1} x \Leftrightarrow x = \sin y$ where y is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$y = \cos^{-1} x \Leftrightarrow x = \cos y$ where y is in $[0, \pi]$.

$y = \tan^{-1} x \Leftrightarrow x = \tan y$ where y is in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Logarithmic and Exponential Functions

$y = \log_a x \Leftrightarrow a^y = x$

$\log_a a^x = x$ and $a^{\log_a x} = x$

$\log_a A \cdot B = \log_a A + \log_a B$

$\log_a \frac{A}{B} = \log_a A - \log_a B$

$\log_a x^n = n \log_a x$

$y = \ln x \Leftrightarrow e^y = x$.

$\ln e^x = x$ and $e^{\ln x} = x$

$\ln A \cdot B = \ln A + \ln B$

$\ln \frac{A}{B} = \ln A - \ln B$

$\ln x^n = n \ln x$

$\ln e = 1$ and $\ln 1 = 0$

Integration Summary

$$\int u^n \, dx = \frac{u^{n+1}}{n+1} + C \; n \neq -1$$

$$\int (ax+b)^n \, dx = \frac{1}{a(n+1)}(ax+b)^{(n+1)} + C$$

$$\int \frac{1}{u} \, du = \ln|u| + C$$

$$\int \frac{d}{ax+b} \, dx = \frac{d}{a} \ln|ax+b| + C$$

$$\int \sin u \, du = -\cos u + C$$

$$\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \tan u \, du = \ln|\sec u| + C$$

$$\int \tan(ax+b) \, dx = \frac{1}{a}\ln|\sec(ax+b)| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \cot(ax+b) \, dx = \frac{1}{a}\ln|\sin(ax+b)| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \sec(ax+b) \, dx = \frac{1}{a}\ln|\sec(ax+b) + \tan(ax+b)| + C$$

$$\int \csc u \, du = \ln|\csc u - \cot u| + C$$

$$\int \csc(ax+b) \, dx = \frac{1}{a}\ln|\csc(ax+b) - \cot(ax+b)| + C$$

$$\int \sec^2 u \, du = \tan u + C$$

$$\int \csc^2 u \, du = -\cot u + C$$

$$\int \sec u \tan u \, du = \sec u + C$$

$$\int \csc u \cot u \, du = -\csc u + C$$

$$\int e^u \, du = e^u + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$$

$$\int a^u \, du = \frac{a^u}{\ln a} + C$$

$$\int a^{Ax+b} \, dx = \frac{a^{ax+b}}{A \ln a} + C$$

$$\int \frac{1}{\sqrt{a^2 - u^2}} \, du = \sin^{-1} \frac{u}{a} + C$$

$$\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}} \, du = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$$